Thermoviscoplasticity deduced from enhanced rheological models

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A new ideal body is proposed for representing isotropic hardening. Hence, a rheological model of thermoviscoplasticity may be assembled with linear isotropic and kinematic hardening and nonlinear strain rate sensitivity. The related constitutive equations including the yield function and the flow rule are directly deduced from the kinematics and the stress equilibrium of the rheological network and result in a well–known model. Based on the free energy of the rheological network, the equation of heat conduction is obtained with the dissipative heat source term, driven by plastic deformations.

1 Introduction

In the concept of rheological models, basic elements like springs (ideal elastic), dashpots (ideal viscous) or friction elements (ideal plastic) are assembled to networks for representing complex material behavior [1, 2]. In case of viscoelasticity, the phenomenological constitutive equations are usually deduced directly from the spring–dashpot–networks [2, 3], but in the elastoplastic material theory, the rheological networks are mostly used only to visualize the fundamental structure of the related material model. However, as presented subsequently, similar ways of proceeding as in viscoelasticity are possible to deduce the elastoviscoplastic material equations based on rheological networks including the yield condition and the flow rule.

2 Thermoviscoplastic rheological model and its constitutive equations

Figure 1 presents the rheological model of thermoviscoplasticity with linear isotropic and kinematic hardening and nonlinear strain rate sensitivity. Its thermoelastic contribution comprises a series connection of a thermal strain element (with thermal expansion coefficient α) on the left hand side and a linear spring (stiffness E) in the middle of the network. The viscoplastic part on the right hand side of the rheological model, however, consists of four chains arranged in parallel, each one representing a specific phenomenon of the entire elastoviscoplastic material response. The nonlinear dashpot (parameters $\{\eta, m, D_0\} > 0$) on top of the viscoplastic contribution is necessary to include nonlinear rate dependency. The friction body below is placed to account for the initial yield limit κ_0. Next, the new defined hardening element with the stiffness E_κ is used to model isotropic hardening. Finally, the linear spring (stiffness E_ξ) at the bottom of the viscoplastic arrangement represents kinematic hardening.

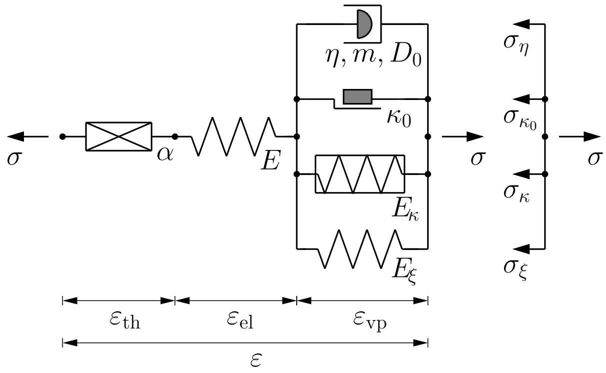

Fig. 1: Rheological model of thermoviscoplasticity with stress and strain decomposition

The stress σ_η of the **nonlinear dashpot** is defined in $(1)_{a,b}$, in which $\operatorname{sgn}(\,)$ denotes the signum function. The stress σ_ξ of the **linear spring** in the **viscoplastic arrangement** — eq. $(1)_c$ — is identical to the internal stress of **kinematic hardening** ξ:

$$\sigma_\eta := |\sigma_\eta| \operatorname{sgn}(\dot\varepsilon_{\rm vp}) \quad , \qquad |\sigma_\eta| := (\eta\,|\dot\varepsilon_{\rm vp}|)^{1/m} D_0 \quad , \qquad \sigma_\xi = E_\xi \varepsilon_{\rm vp} =: \xi \tag{1}$$

If the **friction body** is strained, then its stress σ_{κ_0} is equal to the product of the initial yield limit κ_0 times the loading direction $\operatorname{sgn}(\dot\varepsilon_{\rm vp})$. However, if the absolute value of the stress $|\sigma_{\kappa_0}|$ is less than the yield limit κ_0, no strains evolve in this element:

$$\sigma_{\kappa_0} := \kappa_0 \operatorname{sgn}(\dot\varepsilon_{\rm vp}) \qquad \text{for} \qquad \dot\varepsilon_{\rm vp} \neq 0 \tag{2}$$

$$|\sigma_{\kappa_0}| < \kappa_0 \qquad \text{for} \qquad \dot\varepsilon_{\rm vp} = 0 \tag{3}$$

The novel rheological body for representing isotropic hardening has to respond every strain process with an increasing resistance. Thus, the strength κ of the **hardening element** is defined as the product of the stiffness E_κ times the arclength $\bar\varepsilon_{\rm vp} := \int_0^t |\dot\varepsilon_{\rm vp}|\,{\rm d}\tau$ of the strain in this component, i.e. the internal stress variable κ of **isotropic hardening** reads $\kappa := E_\kappa \bar\varepsilon_{\rm vp}$.

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Moreover, during the loading process, the algebraic sign of the stress σ_κ must be equal to the loading direction of the element: $\mathrm{sgn}(\sigma_\kappa) = \mathrm{sgn}\big(\dot{\varepsilon}_{\mathrm{vp}}\big)$. Hence, similar stress relations as in eqs. (2) and (3) are obtained for the hardening body:

$$\sigma_\kappa := E_\kappa\, \bar{\varepsilon}_{\mathrm{vp}}\, \mathrm{sgn}\big(\dot{\varepsilon}_{\mathrm{vp}}\big) = \kappa\, \mathrm{sgn}\big(\dot{\varepsilon}_{\mathrm{vp}}\big) \qquad \text{for} \quad \dot{\varepsilon}_{\mathrm{vp}} \neq 0 \tag{4}$$

$$|\sigma_\kappa| < E_\kappa\, \bar{\varepsilon}_{\mathrm{vp}} = \kappa \qquad \text{for} \quad \dot{\varepsilon}_{\mathrm{vp}} = 0 \tag{5}$$

The stored energy of the hardening element results as the time integral of its stress power: $\rho\psi_\kappa = \int_0^t \sigma_\kappa\, \dot{\varepsilon}_{\mathrm{vp}}\, \mathrm{d}\tau = \frac{1}{2} E_\kappa \bar{\varepsilon}^2_{\mathrm{vp}}$.

The **kinematics** of the material model is directly obtained from the rheological network as an additive split of the total strain ε into the thermal, elastic and viscoplastic contribution: $\varepsilon = \varepsilon_{\mathrm{th}} + \varepsilon_{\mathrm{el}} + \varepsilon_{\mathrm{vp}}$. The fraction of the thermal strain element is defined as $\varepsilon_{\mathrm{th}} := \alpha(\theta - \theta_0)$. Moreover, an additive decomposition of the total stress $\sigma = E\varepsilon_{\mathrm{el}}$ follows from the rheological model: $\sigma = \sigma_\eta + \sigma_{\kappa_0} + \sigma_\kappa + \sigma_\xi$. Inserting the stress relations $(1)_{a,c}$, (2) and (4) into the **balance equation of stresses** yields:

$$\sigma = \sigma_\eta + \sigma_{\kappa_0} + \sigma_\kappa + \sigma_\xi = |\sigma_\eta|\ \mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}}) + \kappa_0\ \mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}}) + \kappa\ \mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}}) + \xi \tag{6}$$

It is assumed temporarily for eq. (6) that the stress relations (2) and (4) of the friction and the hardening body are valid in any case, which is equivalent to the **tentative hypothesis**: *Only elastoplastic states are possible in the rheological network, i.e. no purely elastic domain exists and $\dot{\varepsilon}_{\mathrm{vp}} \neq 0$ holds for any time*. However, this assumption causes a mathematical contradiction as soon as purely elastic behavior occurs, in which $\dot{\varepsilon}_{\mathrm{vp}} \equiv 0$ is certainly valid. In this manner, in the course of the rearrangements (7) and (8), the condition arises to distinguish between the elastic and the plastic state of the model. Rewriting of (6) leads to

$$\sigma - \xi = |\sigma - \xi|\ \mathrm{sgn}(\sigma - \xi) = \big(|\sigma_\eta| + \kappa_0 + \kappa\big)\ \mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}})\ . \tag{7}$$

Eq. (7) is split up into the absolute value $|\sigma - \xi| = |\sigma_\eta| + \kappa_0 + \kappa$ and the algebraic sign $\mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}}) = \mathrm{sgn}(\sigma - \xi)$, which directly gives the development direction of the viscoplastic strain rate $\dot{\varepsilon}_{\mathrm{vp}}$. The former relation is rearranged with $(1)_b$ as:

$$(\eta\, |\dot{\varepsilon}_{\mathrm{vp}}|)^{1/m}\, D_0 = |\sigma_\eta| = |\sigma - \xi| - (\kappa_0 + \kappa) =: f \tag{8}$$

The left hand side of eq. (8) is strictly positive — cf. $(1)_b$. However, dependent on the actual state of the total and the internal stress variables σ, ξ, κ and κ_0, the function f can have negative values as well. The mathematical contradiction in (8) in the case of $f < 0$ results as expected in consequence of the hypothesis made previously that *exclusively a elastoplastic behavior would exist*. This means, however, for $f < 0$, no plastic loading is possible in the rheological model — instead, a purely elastic step takes place and the stress relations (3) and (5) of the friction and the hardening body are valid in lieu of eqs. (2) and (4). Thus, the well–known **yield function** $f = |\sigma - \xi| - (\kappa_0 + \kappa)$ with the **case distinction** $\{f \leq 0 \Leftrightarrow$ elastic domain / $f > 0 \Leftrightarrow$ viscoplastic domain$\}$ arises in a natural way from the balance of stresses of the rheological network (6). Moreover, eq. (8) provides the constitutive relation $|\dot{\varepsilon}_{\mathrm{vp}}| = \frac{1}{\eta}\,(f/D_0)^m$. Thus, the **flow rule** of the viscoplastic strain $\varepsilon_{\mathrm{vp}}$ results as:

$$\dot{\varepsilon}_{\mathrm{vp}} = |\dot{\varepsilon}_{\mathrm{vp}}|\ \mathrm{sgn}(\dot{\varepsilon}_{\mathrm{vp}}) = 1/\eta\ \langle f/D_0\rangle^m\ \mathrm{sgn}(\sigma - \xi) \tag{9}$$

To ensure $\dot{\varepsilon}_{\mathrm{vp}} = 0$ in (9) during the elastic step $f < 0$, the Macauley–bracket $\langle x\rangle := (x + |x|)/2$ is inserted for the term f/D_0.

The **free energy** ψ of the rheological model is summed up from all contributions of the ideal bodies with energy storage, namely from the thermal strain element ψ_{th}, both linear springs ψ_{el} and ψ_ξ as well as the hardening component ψ_κ, and reads

$$\psi = \psi_{\mathrm{th}} + \psi_{\mathrm{el}} + \psi_\kappa + \psi_\xi = -c_{\mathrm{d}}\,(\theta - \theta_0)^2/(2\theta_0) + \big(E\varepsilon^2_{\mathrm{el}} + E_\kappa \bar{\varepsilon}^2_{\mathrm{vp}} + E_\xi \varepsilon^2_{\mathrm{vp}}\big)/(2\rho)\,. \tag{10}$$

The **equation of heat conduction** results with the **mechanical dissipation power** $\rho\delta_{\mathrm{M}}$, driven by plastic deformations, [3,4]:

$$\rho c_{\mathrm{d}}\, \dot{\theta} = -E\alpha\, \theta \dot{\varepsilon}_{\mathrm{el}} + k\, \mathrm{div}(\mathrm{grad}\, \theta) + \rho b + \rho\delta_{\mathrm{M}} \quad , \qquad \rho\delta_{\mathrm{M}} = (f + \kappa_0)\, |\dot{\varepsilon}_{\mathrm{vp}}| \geq 0 \tag{11}$$

The summands of $\rho\delta_{\mathrm{M}}$ correspond to the fractions of the stress power, applied to the dashpot and the friction body, respectively.

3 Summary and outlook

A rheological model of thermoviscoplasticity is presented, accounting for linear isotropic and kinematic hardening as well as nonlinear rate dependency, and its constitutive equations are deduced directly from the relations of the rheological network.

Moreover, nonlinear isotropic and kinematic hardening as well as an improved description of energy storage and dissipation may be realized by introducing three novel dissipative strain elements into the viscoplastic arrangement of the rheological model in figure 1. The related procedure for deducing the constitutive equations will be published in the forthcoming paper [4].

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