Gerlach, S.; Matzenmiller, A.; Sippel ,W.

Identification of retardation spectra by approximation theory

A numerical method is described which generates the discrete retardation spectrum from the response function of a static experiment. The identification of the retardation spectrum is a nonlinear optimization problem with non-negative constraints on the parameters. The algorithm presented solves the identification task in three steps by using the Tschebyscheff*-approximation and the quadratic optimization method by* Wolfe*. An application to experimental data is given.*

1. Introduction

The retardation function J and the relaxation function G must be determined from experimental data for linear viscoelastic material with the following constitutive equations:

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau)\,\dot\sigma(\tau)d\tau \qquad \sigma(t) = \int_{-\infty}^{t} G(t-\tau)\,\dot\epsilon(\tau)d\tau \tag{1}$$

where $\dot\epsilon$ and $\dot\sigma$ denote time derivatives of the strain and stress history. Especially the relaxation function is needed as the input data for the displacement formulation of the FE-method, if a boundary value problem with a viscoelastic material model should be analysed numerically. Due to the fact that usually only creep data is available from static experiment, a solution scheme has to be proposed, which determines the parameters for the retardation function form the given data by a creep experiment. Afterwards the retardation function must be interconverted into the relaxation function.

The scope of this paper is the fitting of the creep data with an appropriate retardation function. A finite Dirichlet–Prony serie is chosen for the characteristic material function, presenting well the linear viscoelastic behavior in general. In the case of the retardation function it corresponds to the generalized rheological model of Kelvin, for the relaxation function to the generalized Maxwell model.

$$J_N(t) = J_0 + \sum_{\nu=1}^{N} J_\nu \left(1 - \mathrm{e}^{-\frac{t}{\tau_\nu}}\right) \qquad G_N(t) = G_0 + \sum_{\nu=1}^{N} G_\nu \mathrm{e}^{-\frac{t}{\theta_\nu}} \tag{2}$$

The unknown material parameters J_ν and τ_ν, denoting retardation coefficients and retardation times, represent the discrete retardation spectrum. For physically realistic materials these parameters are positive. Therefore, the fitting algorithm has to take into account these constraints ($J_\nu \geq 0$, $\tau_\nu \geq 0$).

2. Numerical method of the parameter identification

The identification of the retardation spectrum is a nonlinear optimization problem with non-negative constraints. The application of the discrete least square method leads to a set of nonlinear normal equations. They involve the difficulties that the convergence to the minimal solution and the compliances with the constraints is not guaranteed. The approximation task is divided into three substeps:

1. Approximation of the creep data by a qualified tendency function J_T in order to allow the approximation by the Dirichlet–Prony series by means of the Tschebyscheff–method. Here, a power law ($J_T = \gamma\, t^\alpha$) is introduced as a tendency function for demonstration purposes.
2. Tschebyscheff–approximation of the tendency function J_T by the model function J_N in eq. (2). With the maximum norm in the approximation of the function J_T the Tschebyscheff–approximation is obtained, which converges to the minimal solution for a nonlinear exponential approximation [2]. By rearranging the Tschebyscheff approach, a set of nonlinear equations is obtained:

$$J_N(t_\mu) + (-1)^\mu\,\lambda = J_T(t_\mu) \tag{3}$$

with J_N according to eq. 2a and unkowns J_0, J_ν, t_ν and λ for $\nu = 1, 2, \ldots, N$ and given N.

The solution of equation (3) leads to the error function $e = J_T - J_N$, which is an alternating function with absolute value λ at the time points t_1, t_2, t_3, ... as shown in figure 1. For the determination of this error function a system of nonlinear equations is solved by an iterative process, which needs starting values for the material parameters and a set of time points for the first alternating error function.

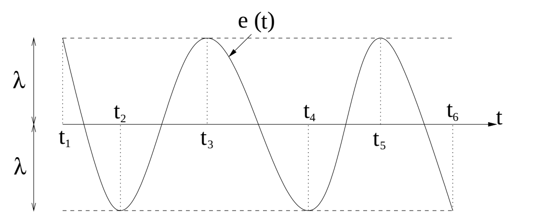

Fig.1: Error function after iteration method

3. Redetermination of the retardation coefficients by fitting the model function J_N to the creep data:
   In consideration of the constraints and the calculated retardation times the retardation coefficients are fitted to the creep data with the least square method. The retardation times are kept the same as found in step 2. The inequality constraints are transformed into equations by introducing slack variables. By using the LAGRANGE multiplier method, the necessary condition for the minimal solution leads to a set of linear and simple nonlinear algebraic equations. This set can be solved by a modified simplex method (WOLFE's method - see [1]). The necessary objective function (6) of the simplex method is introduced by expanding the system of linear equations with artical variables $\mathbf{v}$ - see eq.(4), where $\mathbf{J}$ and $\mathbf{u}$ denote the vectors of the unkown retardation coefficients and the Lagrange multipliers.

The solution of the minimization task is found, if all artificial variables are equal to zero. The fulfillment of the nonlinear equation (5) is guaranteed by modifying the usual rules for the simplex method. In practise this means that at least one variable of the pair (u_0, J_0),(u_1, J_1) ... (u_N, J_N) is not allowed to be a basisvariable ($\neq 0$).

$$\mathbf{A^T \hat{J}} = \mathbf{A^T A\, J} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{u} + \mathbf{v} \qquad (4)$$

$$\mathbf{u^T J} = 0 \qquad (5)$$

$$z = \sum_{k=0}^{N} v_k \to \text{Min. !} \qquad (6)$$

3. Example

The method presented is applied to creep data for a unidirectional composite. The left figure shows that the description of the experimental data with a power law is only a crude approximation, however, for the purpose of a tendency function it is adequate. The two other pictures show that an increasing number of exponential terms improve the approximation of the creep data.

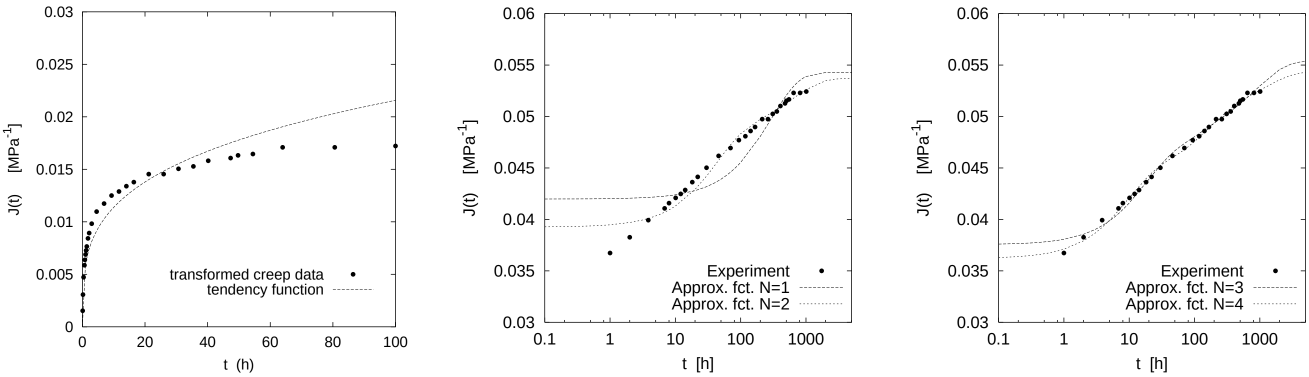

Fig.2: Approximation of the creep data by a power law (a) and by Dirichlet–Prony series with different number of exponentials $N = 1, 2, 3$ or 4 (b),(c)

4. References

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Addresses: DIPL.-ING. SEBASTIAN GERLACH, PROF. DR.-ING. ANTON MATZENMILLER,
Institute of Mechanics, University Gh Kassel, 34109 Kassel, Germany
PROF.DR. WILHELM SIPPEL, Department of Mathematics, University Gh Kassel, 34109 Kassel