A RETURN MAPPING ALGORITHM FOR ISOTROPIC ELASTOPLASTICITY

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SUMMARY

A return mapping algorithm is presented for the numerical time integration of the constitutive equations for elastoplasticity with isotropic yield surfaces, constructed from all three invariants of the stress tensor. Based on the first-order backward Euler difference formula (BDF), the governing equations for the stresses are solved in the space of the invariants and the discretized persistence parameter. The stresses are recovered afterwards. The solution concept is applied to a pressure-independent yield function, expressed in terms of the second and third invariant of the stress tensor. The numerical performance of the method is demonstrated with two examples.

1. INTRODUCTION

Effective time integration algorithms for general isotropic yield criteria in elastoplasticity are still under investigation. In order to advance the solution in time, the elastic predictor and plastic corrector scheme, based on the elastoplastic operator split, is a widely accepted method for time integration of the constitutive equations in flow theory. The computational attractiveness of these one-step schemes is cost efficiency with regard to the necessary number of floating point operations. This is achieved by means of simple function evaluations for the solution of the elastic problem in the 'predicting step' and an efficient strategy for solving the implicit algebraic equations of the plastic 'correcting' step. Despite major computational success with the so-called 'radial return mapping algorithm' for J_{2}-plasticity see Reference 9, a similar solution strategy has not been proposed for general isotropic yield surfaces, where comparable advantage of the isotropic properties can be made as in the case of the Prandtl–Reuss flow theory for the solution of the governing algebraic equations.

2. GOVERNING EQUATIONS AND DISCRETIZATION

2.1. Constitutive equations

The constitutive equations of classical plasticity are derived from the elastic, \( W \), and plastic potential functions. For isotropic material models, the potentials depend on the invariants of the stress, \( \sigma \), or strain tensor, \( e \). It is advantageous to separate the spherical part, \( p \), from the stress deviator, \( S \),

\[
S = \sigma - p I \quad \text{with} \quad p = \frac{1}{3} \text{tr} \sigma, \quad I = [\delta_{ij}] \quad (1)
\]

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and make use of the three principal invariants, defined as

\begin{align}
I_1 &= \text{tr} \sigma \\
J_2 &= \frac{3}{2} \text{tr} (\Sigma) \\
J_3 &= \frac{3}{4} \text{tr} (\Sigma \Sigma)
\end{align}

where \( I \) is the second-order identity tensor, \( \delta_{ij} \) represents the Kronecker symbol and \( \text{tr} \) stands for the 'trace operator'. The transformation to other invariants is straightforward; however, in subsequent developments equation (2) has favourable properties. The strain tensor may be split into its elastic and plastic parts:

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

It is assumed that linear elasticity with the constitutive tensor \( D \) governs the elastic response between stresses and elastic strains \( (\sigma - \varepsilon^p) \). Accordingly, the constitutive relationship at a fixed state of plastic strains is

\[ \sigma = \frac{\partial W}{\partial \varepsilon} = D (\varepsilon - \varepsilon^p) \quad (3a) \]

Of special interest for the numerical scheme, investigated subsequently, will be the case, for which the elastic part of the material response is isotropic:

\[ D = K I \otimes I + 2\mu (I - \frac{1}{3} I \otimes I) \quad (3b) \]

with

\[ I = \frac{1}{2} \left[ \delta_{ij} \delta_{ji} + \delta_{ij} \delta_{ji} \right] \quad \text{and} \quad I \otimes I = [\delta_{ij} \delta_{kl}] \]

where \( I \) is a symmetric rank-four identity tensor, ensuring the minor symmetry \( D_{ijkl} = D_{jikl} \), and \( K, \mu \) are the elasticity constants for the bulk and the shear modulus, respectively.

The projection of the stress tensor \( \sigma \) onto its deviatoric tensor \( S \) may also be performed by the same operator \( (I - \frac{1}{3} I \otimes I) \). However, the symmetry of the stress tensor allows the projection operator \( P \) for the deviatoric \( S = P \sigma \) to be constructed by

\[ P = [\delta_{ij} \delta_{kl} - \frac{1}{3} \delta_{ij} \delta_{kl}] \]

It can be shown that the projection operator \( P \) is symmetric and idempotent.

Furthermore, the normality rule holds for the plastic flow \( \dot{\varepsilon}^p \), i.e. a plastic potential is identified with the yield criterion \( F \), which is expressed as a function of the previously defined invariants:

\[ F = f(I_1, J_2, J_3) = 0 \quad (4) \]

The flow rule is governed by

\[ \dot{\varepsilon}^p = \gamma \frac{\partial f}{\partial \sigma} \quad (5a) \]

For simplicity, no further internal variables are introduced to describe hardening, since the main thrust here is treating yield functions, which include \( J_3 \). The flow rule for an isotropic yield criterion specializes to

\[ \dot{\varepsilon}^p = \gamma \left[ \frac{\partial f}{\partial I_1} I + \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3} J_2 I) \right] \quad (5b) \]
Equations (1)-(5b) form a system of differential-algebraic equations, for which the preservation of the yield criterion during the advancement of the solution is of major concern in the numerical scheme.

2.2. Time discretization and transformation into space of invariants

The time discretization of the flow equations by means of the Euler backward difference formula provides an unconditionally stable finite difference expression for advancing the inelastic strains from time \( t_n \) to \( t_{n+1} \):

\[
\varepsilon_{e,n+1}^p = \varepsilon_e^p + \left. \lambda \left[ \frac{\partial f}{\partial J_1} 1 + \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3} J_2 1) \right] \right|_{t_{n+1}}
\]

with \( \lambda = \gamma_{e,n+1} - \gamma_n \).

For a given total strain, \( \varepsilon_{e,n+1} \), at time \( t_{n+1} \) the plastic strains \( \varepsilon_{e,n+1}^p \) may be eliminated by using the elasticity law in equations (3a) and (3b). The resulting equation provides an expression for the stress state \( \sigma_{e,n+1} \) at the end of the current time step:

\[
\sigma_{e,n+1} = D(\varepsilon_{e,n+1} - \varepsilon_e^p) - \lambda D \left[ \frac{\partial f}{\partial J_1} 1 + \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3} J_2 1) \right] |_{t_{n+1}}
\]

(6)

This relationship motivates the definition of a trial stress \( \sigma^\ast \) in computational plasticity (also called the elastic predictor), which is given by

\[
\sigma^\ast = D(\varepsilon_{e,n+1} - \varepsilon_e^p)
\]

(7)

for the Euler backward difference scheme, defined above.

Remark. The trial stress \( \sigma^\ast \) may also be regarded as the predictor for the more general class of standard predictor-corrector methods. The special predictor, based on equation (7), may be interpreted as the stress state, resulting from an entirely elastic response of the material within the current time increment. The change of the plastic state completely takes place in the corrector step—often denoted as a plastic corrector. Because of the additive nature of the elastic and plastic strains in the mechanical model, the elastoplastic behaviour can be split into two parts for the construction of the numerical time integration scheme. The elastic part is solved in the 'predictor step' and the plastic contribution is accommodated during the 'corrector step'.

Based on the Euler backward difference formula for numerical time integration of plastic flow, the set of governing equations is given by combining equations (7) and (6):

\[
\sigma_{e,n+1} = \sigma^\ast - \lambda D \left[ \frac{\partial f}{\partial J_1} 1 + \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3} J_2 1) \right] |_{t_{n+1}}
\]

(8)

The isotropic elasticity tensor in equation (3b) is substituted into equation (8) to give

\[
\sigma^\ast = \sigma_{e,n+1} + \left\{ K \lambda \left[ \frac{\partial f}{\partial J_1} 1 + 2\mu \lambda \left[ \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3} J_2 1) \right] \right] \right\} |_{t_{n+1}}
\]

(9)

For the given trial stress \( \sigma^\ast \), equation (9) together with the yield criterion determine the six unknown components of the stress tensor and the increment of the plastic multiplier \( \lambda \) for the current time step. However, for isotropic yield functions it is possible to transform the set of equations, given above, from the space of stress components onto the reduced space of its invariants. The trial stress tensor in equation (9) is mapped into its three invariants \( I_1^\ast, J_2^\ast, J_3^\ast \). The resulting set of equations is solved in terms of the invariants of the stress tensor \( \sigma_{e,n+1} \).
In the following the notation is simplified by omitting the index \( n + 1 \), since only the stress components at time \( t_{n+1} \) appear in the governing equations besides the trial stresses. In addition, the shorthand notation for \( \tilde{f}_1 = \frac{\partial f}{\partial J_1} \), \( \tilde{f}_2 = \frac{\partial f}{\partial J_2} \), \( \tilde{f}_3 = \frac{\partial f}{\partial J_3} \) and \( \tilde{f}_4 = \frac{\partial f}{\partial J_4} \) is used for the partial derivatives at time \( t_{n+1} \). After the trace operator is applied to equation (9), the stress tensor \( \sigma_{n+1} \) is split into its spherical and deviatoric parts:

\[
p = \frac{1}{3} \text{tr} \sigma = p^* - K J_1^S \quad \text{and} \quad p^* = \frac{1}{3} \text{tr} \sigma^* = \frac{1}{3} f_1^S
\]

(10a)

Also needed are

\[
S^* = (1 + 2\mu J_2^S)S + 2\mu J_1^S (SS - \frac{1}{3} J_2 1)
\]

(11a)

and similarly \( S^* S^* S^* \).

By using the Cayley–Hamilton theorem

\[
SSS = J_2 S + J_3 1
\]

(12)

the two moment invariants \( J_2^S \) and \( J_3^S \) are formed from the deviator \( S^* \) in equation (11a) of the trial stress \( \sigma^* \):

\[
J_2^S = \frac{1}{3} \text{tr} (S^* S^*) = [(1 + 2\mu J_2^S)^2 + (2\mu J_3^S)^2 \frac{1}{3} J_2] J_2
+ (1 + 2\mu J_3^S)^2 6\mu J_1 J_3
\]

(10b)

\[
J_3^S = \frac{1}{3} \text{tr} (S^* S^* S^*) = (1 + 2\mu J_3^S)^3 J_3 + (1 + 2\mu J_3^S)^2 (2\mu J_5^S) \frac{1}{3} J_2^S
+ (1 + 2\mu J_3^S)^3 J_3 + (2\mu J_5^S)^2 J_2 J_3 + (\mu J_5^S)^3 (J_3^2 - \frac{1}{2} J_2^2)
\]

(10c)

The next objective is to solve the three equations (10a)–(10c) and the yield criterion in equation (4) for the invariants \( I_1, J_2 \) and \( J_3 \) of the stress state and the plastic multiplier increment \( \lambda \) at the current time step \( t_{n+1} \). Except for special cases, iterative methods are used to solve the set of non-linear equations.

3. EVALUATION OF THE STRESS STATE

3.1. Solution in the space of invariants

Successful numerical solutions, based on the elastic predictor, are only obtained for large time steps as long as the elastic range, bounded by the yield surface, is convex and connected. Then the solution is unique in the strain-controlled case, where the strain rate—i.e. the strain increment after converting rates into increments—is given.

Practical yield criteria are frequently formulated in the deviatoric plane in terms of polar co-ordinates. Typically, the radial co-ordinate \( \rho \) is chosen as the distance of the stress state to the hydrostatic pressure axis in the space of the principal stress components:

\[
\rho = \sqrt{2 J_2} = ||S||
\]

(13a)

where \( || \cdot || \) is the Euclidean norm of the stress deviator.
The angle co-ordinate $\theta$ is usually identified with the 'Lode angle' $\varphi = \theta/3$. For notational convenience the variable $\zeta$ is introduced for $\cos \theta$:

$$\zeta = \cos \theta = \frac{J_3/2}{(J_3/3)^{1/2}}$$  \hspace{1cm} (13b)

The partial derivatives $\hat{f}_j$ of the yield function $f(I_1, J_2, J_3) = f(p, \rho, \zeta) = 0$ are transformed by means of the Jacobian $\hat{e}(p, \rho, \zeta)\hat{e}(I_1, J_2, J_3)$ into the partial derivatives $[f_1, f_2, f_3] = [\hat{e}_f/\hat{e}p, \hat{e}_f/\hat{e}\rho, \hat{e}_f/\hat{e}\zeta]$ with respect to the physically more meaningful invariants $p, \rho$ and $\zeta$:

$$\begin{bmatrix}
\hat{f}_1 \\
\hat{f}_2 \\
\hat{f}_3
\end{bmatrix} =
\begin{bmatrix}
1/3 & 0 & 0 \\
0 & 1/\rho & -3\zeta/\rho^2 \\
0 & 0 & 3\sqrt{6}/\rho^3
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}$$  \hspace{1cm} (14)

Equations (10a)–(10c) and the yield criterion are written in implicit form next. After defining

$$\Lambda = 2\mu$$

$$a = \delta(p, \rho, \zeta, \Lambda) = \rho + \Lambda \left( f_2 - \frac{3\zeta}{\rho} f_3 \right)$$  \hspace{1cm} (15)

$$b = \hat{b}(p, \rho, \zeta, \Lambda) = \frac{3\Lambda}{\rho} f_3$$

the set of governing equations is cast into matrix notation:

$$R =
\begin{bmatrix}
R_1 \\
R_2 \\
R_3 \\
R_4
\end{bmatrix} =
\begin{bmatrix}
1 - 3p - (K/2\mu)\Lambda f_1 \\
2J_2^* - (a^2 + 2ab\zeta + b^2) \\
3\sqrt{6}J_3^* - \left[ a^3\zeta + 3a^2b + 3ab^2\zeta + b^3(2\zeta^2 - 1) \right] \\
f(p, \rho, \zeta, \Lambda)
\end{bmatrix} = 0$$  \hspace{1cm} (16)

Newton’s method is a standard technique to solve the system of non-linear, algebraic equations $R(x) = 0$ in terms of the unknowns $x = [p, \rho, \zeta, \Lambda]^T$. The resulting equations for the $j$th iterate

$$R|_{x^{(0)} + \tau \Delta x^{(j)}} + \frac{\partial R}{\partial x}|_{x^{(0)}} \Delta x^{(j)} = 0$$  \hspace{1cm} (17a)

are solved by starting the recurrence scheme with the initial values $x^{(0)} = [p^*, \rho^*, \zeta^*, 0]^T$ equal to the invariants $p^*$, $\rho^*$ and $\zeta^*$ of tensor for the trial stress state $\sigma^*$. 

Remarks

(a) The iteration matrix $\partial R/\partial x$ for the new unknowns is found to be better conditioned with respect to inversion, since the transformation to the new unknowns includes a 'scaling' procedure. In fact, the new unknowns have either the dimension of stress or they are dimensionless.

(b) The solution algorithm is easily adapted to modifications of the yield criterion. Only the last equation for $R_4 = 0$ and the auxiliary functions $a = \delta(p, \rho, \zeta, \Lambda)$, $b = \hat{b}(p, \rho, \zeta, \Lambda)$ change.

(c) The numerical costs significantly reduce, if the yield criterion can be written as the sum of two functions $f(p, \rho, \zeta, \Lambda) = f_1(p) + f_2(p, \rho, \zeta, \Lambda) = 0$, such that the pressure and deviatoric...
parts are additively decomposed. The criteria of Mohr–Coulomb, Drucker–Prager or Mises–Schleicher fall into this class. Under these circumstances the mixed partial derivatives \( \partial/\partial(\rho, \xi, \varphi) \) vanish and one Gauss-elimination step suffices to limit the number of operations to the inversion of a rank-three submatrix with additional back substitution for solving equation (17a):

\[
[A_{ij}] = \begin{bmatrix}
\partial R_1 & \partial R_2 & \partial R_3 \\
\partial R_2 & \partial R_3 & \partial R_4 \\
\partial R_3 & \partial R_4 & \partial \bar{R}_{44}
\end{bmatrix}
\]

with \( \bar{R}_{44} = -R_4 R_1 R_2 R_4 \) (17b)

(d) Whenever the third invariant \( J_3 \) is missing in the yield criterion, the auxiliary function \( b \) vanishes in equation (16) and the Lode angle for the final stress state is equal to the one for the trial stress state. The unknown \( \xi \) follows from the equations for \( R_2 \) and \( R_3 \) as

\[
\xi = \xi^* = 3/6 J^*_3 / (2 J^*_2)^{1/2}
\]

The additional definitions \( p^* = \frac{1}{2} J^*_2 \) and \( \rho^* = \sqrt{2 J^*_2} \) besides \( \xi^* \) will become helpful next.

(e) On the von Mises yield criterion: The simple case of the von Mises yield criterion \( f_{\text{M}} = \rho - R_0 = 0 \) with material constant \( R_0 \) has \( a = \rho + \Lambda \) and \( b = 0 \). The set of governing equations in (16) leads to the ‘radial return mapping’ algorithm:

\[
\begin{align*}
R_1 &= J^*_2 - 3p = 0 \\
R_2 &= 2J^*_2 - (\rho + \Lambda)^2 = 0 \\
R_3 &= 3\sqrt{6} J^*_3 - (\rho + \Lambda)^2 \xi = 0 \\
R_4 &= \rho - R_0 = 0
\end{align*}
\]

Substitution of the fourth equation into the second and then into the third provides the unique solution under the restriction that \( \Lambda \geq 0 \):

\[
\begin{align*}
p &= \frac{1}{2} J^*_2 \\
\rho &= R_0 \\
\xi &= 3/6 J^*_3 / (2 J^*_2)^{1/2} = \xi^* \\
\Lambda &= f^*_M
\end{align*}
\]

with definition \( f^*_M = \rho^* - R_0 \). Earlier use of these relationships was made in References 2 and 3.

(f) On the Drucker–Prager yield criterion, the yield criterion \( f_{\text{DP}} = 2p + \rho - c_k = 0 \), attributed to Drucker and Prager, with the material constants \( a \) and \( c_k \) is another special case, for which a closed-form solution in terms of the unknown invariants of the stress state is available. Noting as before that \( a = \rho + \Lambda \) and \( b = 0 \), the governing equations become

\[
\begin{align*}
R_1 &= J^*_2 - 3p - K / 2p \Lambda a = 0 \\
R_2 &= 2J^*_2 - (\rho + \Lambda)^2 = 0
\end{align*}
\]
\[ R_3 = 3\sqrt{6} J_2^* - (\rho + \Lambda) \xi = 0 \]
\[ R_4 = zp + \rho - c_4 = 0 \]

Their solution is

\[ p = p^* - \frac{K_2}{6\mu} \frac{f_{0p}^*}{1 + \frac{K_2}{6\mu}} \]
\[ \rho = p^* - \frac{1}{K_2} \frac{f_{0p}^*}{6\mu} \]
\[ \xi = \xi^* \]
\[ \Lambda = \frac{1}{1 + \frac{K_2}{6\mu}} f_{0p}^* \]

where \( f_{0p}^* = zp^* + \rho^* - c_4 \).

Based on physical arguments, both \( \rho \) and \( \Lambda \) must be positive, which implies that the solution is unique under strain control. As for the von Mises criterion the Lode angle of the final stress state is already determined by the trial stress tensor. Essentially the same result was given in Reference 5. If the 'return' is to the singular point on the Drucker–Prager yield surface ('tip of the cone'), the second invariant \( J_2 \) vanishes and the final stress state is purely hydrostatic.

3.2. Recovery of stresses

The stress state \( \sigma_{e+1} \) is recovered from the stress deviator \( S \) by means of equation (11a) and the spherical part in equation (10a) after the multiplier \( \Lambda \) as well as the invariants \( p, \rho \) and \( \xi \) have been evaluated. Instead of directly solving the system of non-linear equations in (11a) for the components of the deviatoric stress, it is simplified by referring to equation (11b). The Cayley–Hamilton theorem of equation (12) is applied to equation (11b) before the change of variables according to equations (13a), (13b) and (14) is performed. By employing the definitions of equation (15), the following two sets of linear equations are derived from equations (11a) and (11b):

\[ \frac{a}{\rho} S + \sqrt{\frac{6b}{\rho^2}} SS = S^* + \frac{\sqrt{6b}}{3} I \]
\[ \sqrt{\frac{6b}{3\rho}} (a + b\xi) S + \frac{a^2 - b^2}{\rho^2} SS = S^* S^* - \frac{2}{3} b(a\xi + b) I \]

where \( S \) and \( SS \) are considered as the two unknowns. From these two equations the deviatoric stress \( S \) at time \( t_{e+1} \) is obtained:

\[ S = \frac{\rho}{a^3 - 3ab^2 - 2b^3 \xi} \left[ \sqrt{\frac{6}{3}} b \left( a^2 + 2ab \xi + b^2 \right) I + (a^2 - b^2) S^* - \sqrt{6b} S^* S^* \right] \quad (18) \]
Remarks:

(i) Whenever \( \rho = 0 \), the stress deviator vanishes, \( S = 0 \).

(ii) In the case of the von Mises criterion the well-known result

\[
S = \frac{R_0}{R_0 + \lambda} S^* = \frac{R_0}{\lambda} \frac{\sqrt{\text{tr}(S^* S^*)}}{\sqrt{\text{tr}(S S^*)}} S^* = \left(1 - \frac{f_{\alpha}^*}{\rho^*}\right) S^*
\]

is recovered, where \( f_{\alpha}^* \) and \( \rho^* \) are defined as before.

(iii) The stress deviator for the Drucker–Prager yield criterion is computed from

\[
S = \left[ 1 - \frac{f_{\alpha}^*}{\left(1 + \frac{K}{3\mu} \rho^*\right)} \right] S^*
\]

where the previous definitions for \( f_{\alpha}^* \), \( \rho^* \) and \( \rho^* \) hold.

(iv) In general, if the yield criterion is independent of \( J_3 \), then \( b = 0 \) and the stress deviator simplifies to

\[
S = \frac{\rho}{\rho^*} S^*
\]

4. ALGORITHMIC TANGENT

The discretized form of the non-linear momentum balance equations is frequently solved by the standard Newton method, for which the linearization of the stresses in terms of the strains is required. The update of the stresses and inelastic strains is performed with respect to the previous equilibrium state in order to avoid spurious yielding between the out-of-balance states at intermediate configurations. The importance of the algorithmic tangent, obtained by linearization of the constitutive algorithm, has been pointed out first in References 6 and 7 and will be given next for the case of isotropic yield functions. The time derivatives of equation (3a)

\[
\dot{\epsilon}_{\alpha_{\alpha}} = \mathbf{D}^{-1} \frac{\partial f}{\partial \dot{\epsilon}_{\alpha_{\alpha}}} + \dot{\epsilon}_{\alpha} \dot{\epsilon}_{\alpha_{\alpha}}
\]

and of the discretized version of the inelastic strains,

\[
\dot{\epsilon}^e_{\alpha_{\alpha}} = \dot{\epsilon}_{\alpha_{\alpha}} + \lambda \frac{\partial f}{\partial \dot{\epsilon}_{\alpha_{\alpha}}} \dot{\epsilon}_{\alpha_{\alpha}}
\]

at time \( \tau_{n+1} \),

\[
\dot{\epsilon}_{\alpha_{\alpha}} = \lambda \frac{\partial f}{\partial \dot{\dot{\epsilon}}_{\alpha_{\alpha}}} + \lambda \frac{\partial^2 f}{\partial \dot{\epsilon}_{\alpha_{\alpha}}^2} \dot{\epsilon}_{\alpha_{\alpha}}
\]

are combined and written together with the consistency condition, \( \tilde{f} = 0 \), in matrix form:

\[
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\sigma}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{H} & \frac{\partial f}{\partial \dot{\epsilon}} \\
\frac{\partial^2 f}{\partial \dot{\epsilon} \partial \dot{\sigma}} & \lambda
\end{bmatrix}
\begin{bmatrix}
\dot{\epsilon} \\
\dot{\sigma}
\end{bmatrix}
\text{with } \mathbf{H} = \mathbf{D}^{-1} + \lambda \frac{\partial^2 f}{\partial \dot{\epsilon}_{\alpha_{\alpha}}^2}
\]

(19)

The \( \lambda \)-term added to the compliance tensor in \( \mathbf{H} \) is what distinguishes the algorithmic tangent from the tangent for continuum constitutive equations. For isotropic yield criteria \( f(I_1, J_2, J_3) \), the Hessian can be expressed as in the form

\[
\frac{\partial^2 f}{\partial \dot{\epsilon} \partial \dot{\sigma}} = \tilde{f}_1 \mathbf{P} + \tilde{f}_3 \mathbf{J}_{3\text{eq}} + \mathbf{S} \mathbf{r}^T \mathbf{S}^+ \mathbf{r}
\]

(20)
with tensors \( \Pi \) and \( P = \hat{\varepsilon}^2 J_2 / \hat{\varepsilon} \sigma \), as defined in Section 2.1:

\[
\begin{align*}
\tau &= \frac{\hat{\varepsilon} J_3}{\hat{\varepsilon} \sigma} = SS - \frac{2}{3} J_2 \mathbf{I} \\
J_{\text{sym}} &= \frac{\hat{\varepsilon} J_2}{\hat{\varepsilon} \sigma \hat{\varepsilon} \sigma} = [S_{ij} \delta_{ij} + \delta_{ii} S_{ij} - \frac{1}{2} (S_{ij} \delta_{ii} + \delta_{ij} S_{ii})]
\end{align*}
\]

and

\[
\begin{align*}
\hat{f}_{11} &= \frac{\partial \hat{f}}{\partial \hat{s}_{11}}, \quad \hat{f}_{12} = \frac{\partial \hat{f}}{\partial \hat{s}_{12}}, \quad \hat{f}_{13} = \frac{\partial \hat{f}}{\partial \hat{s}_{13}}, \quad \text{etc.}
\end{align*}
\]

The last term in equation (20) is understood to be expanded using matrix convention with tensor operations inserted as shown, e.g., \( \mathbf{1} \otimes \hat{f}_j, \Pi \Pi = \hat{f}_{ij} \mathbf{1} \otimes \mathbf{1} \).

The matrix \( H \) is invertible if \( D^{-1} \) is positive-definite and the yield surface is convex. As an outcome of block elimination in equation (19)

\[
\begin{bmatrix}
\mathbf{I} & \mathbf{0} \\
-\frac{\partial f^*}{\partial \sigma} \mathbf{H}^{-1} & \mathbf{1}
\end{bmatrix}
\begin{bmatrix}
\hat{\varepsilon} \\
\hat{\lambda}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial \hat{f}}{\partial \sigma} \\
-\frac{\partial f^*}{\partial \sigma} \mathbf{H}^{-1} \frac{\partial \hat{f}}{\partial \sigma}
\end{bmatrix}
\]

the Schur complement

\[
\frac{\partial f^*}{\partial \sigma} \mathbf{H}^{-1} \frac{\partial \hat{f}}{\partial \sigma}
\]

is recovered. It is negative if the pivot block \( H \) is positive-definite. Equation (21) provides expressions for

\[
\hat{\lambda} = \frac{1}{\frac{\partial f^*}{\partial \sigma} \mathbf{H}^{-1} \frac{\partial \hat{f}}{\partial \sigma}} \frac{\partial f^*}{\partial \sigma} \mathbf{H}^{-1} \hat{\varepsilon}
\]

and the algorithmic tangent stiffness, which is obtained by back substitution:

\[
\hat{\sigma} = \mathbf{H}^{-1} \left[ \Pi - \frac{\partial \hat{f}}{\partial \sigma} \frac{\partial \hat{s}_{11}}{\partial \sigma} \mathbf{H}^{-1} \frac{\partial \hat{f}}{\partial \sigma} \right] \hat{\varepsilon}
\]

5. APPLICATION TO PRESSURE-INDEPENDENT YIELDING

As an example, the proposed solution method is applied to a pressure-independent isotropic yield surface \( f(\rho, \theta) = 0 \), constructed by means of the two moment invariants:

\[
f(\rho, \theta) = \rho (1 + \beta \cos \theta) - \sqrt{\frac{3}{2}} c_4 = 0
\]

For \( ||\beta|| \leq \frac{1}{2} \) the criterion is convex and specializes to the von Mises yield criterion if \( \beta = 0 \). Its yield curve in the \( \pi \)-plane is given in Figure 1. Since the yield criterion is an even function of the third invariant, it is clearly symmetric with respect to the sign of the third invariant, which itself depends on the sign of the deviator components. Symmetry with respect to the axis of the
principal stress deviators $S_1, S_2$ and $S_3$ results as a necessary consequence. Invariance with respect to interchanging co-ordinate axis follows from isotropy.

In terms of the unknowns, introduced earlier, the auxiliary functions $a = p + \Lambda (1 - 2\beta \xi)$ and $b = 3A\beta$ complete the equations for $R_3$ and $R_3$ of the set (16). The chosen yield criterion in equation (22) is a special case of equation (17b). Since $R_2 = \tilde{K}_2(a, b, \xi)$ and $R_3 = \tilde{K}_3(a, b, \xi)$ only depend on the auxiliary functions $a(p, \xi, \Lambda)$ and $b(p, \xi, \Lambda)$ besides the unknown $\xi$, the chain rule of differentiation is applied in order to derive the iteration matrix for the Newton–Raphson method as outlined in equation (17a). The basic structure of the governing equations is preserved:

$$\frac{\partial \mathbf{R}}{\partial \mathbf{x}} \Delta \mathbf{x} = \begin{bmatrix} \frac{\partial R_2}{\partial a} & \frac{\partial R_2}{\partial b} & 0 \\ \frac{\partial R_3}{\partial a} & \frac{\partial R_3}{\partial b} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial a}{\partial \rho} & \frac{\partial a}{\partial \xi} & \frac{\partial a}{\partial \Lambda} \\ \frac{\partial b}{\partial \rho} & \frac{\partial b}{\partial \xi} & \frac{\partial b}{\partial \Lambda} \\ \frac{\partial f}{\partial \rho} & \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \Lambda} \end{bmatrix} \begin{bmatrix} \Delta \rho \\ \Delta \xi \\ \Delta \Lambda \end{bmatrix} + \begin{bmatrix} \frac{\partial R_2}{\partial \xi} \\ \frac{\partial R_3}{\partial \xi} \\ 0 \end{bmatrix} \Delta \xi = 0 \quad (23)$$

Only the partial derivatives of $a$ and $b$ with respect to $\rho$ and $\xi$ change besides the yield function $f$ for a new yield criterion:

$$\frac{\partial a}{\partial \rho} = 1 + \Lambda \left( f_{32} + \frac{3\xi}{\rho} f_{5} - \frac{3\xi}{\rho} f_{33} \right)$$

$$\frac{\partial a}{\partial \xi} = \Lambda \left( f_{32} - \frac{3\xi}{\rho} f_{5} - \frac{3\xi}{\rho} f_{33} \right)$$

$$\frac{\partial a}{\partial \Lambda} = f_{5} - \frac{3\xi}{\rho} f_{5}$$

and

$$\frac{\partial b}{\partial \rho} = \frac{3\Lambda}{\rho} \left( f_{32} - \frac{1}{\rho} f_{5} \right)$$
\[
\frac{\partial b}{\partial \xi} = 3 \frac{\lambda}{\rho} f_{23}, \\
\frac{\partial b}{\partial \lambda} = 3 \frac{1}{\rho} f_3
\]

where
\[
f_{22} = \frac{\partial^2 f}{\partial \rho \partial \rho}, \quad f_{23} = \frac{\partial^2 f}{\partial \rho \partial \xi}, \quad f_{32} = \frac{\partial^2 f}{\partial \xi \partial \rho}, \quad f_{33} = \frac{\partial^2 f}{\partial \xi \partial \xi}
\]

The partial derivatives of \( R_1 \) and \( R_2 \) easily follow from equations (16b) and (16c), since only polynomials are involved. The same holds for \( f \) as specified in equation (22).

Since the yield surface is defined in terms of the invariants \( \rho \) and \( \xi \), the flow vector and the Hessian matrix for the tangent stiffness follow as
\[
\frac{\partial f}{\partial \sigma} = \frac{\partial f}{\partial \rho} \mathbf{n} + \frac{\partial f}{\partial \xi} \mathbf{j}
\]

with
\[
\mathbf{n} = \frac{\partial \rho}{\partial \sigma} = 1 \quad \mathbf{S} = \frac{\mathbf{S}}{\mathbf{S}} \quad \text{and} \quad \mathbf{j} = \frac{\partial \xi}{\partial \sigma} = 3 \left( \frac{\sqrt{6} \frac{\partial J_3}{\partial \sigma}}{\rho^3} \mathbf{c} - \mathbf{e} \right)
\]

The Hessian of the yield function \( f \) is
\[
\frac{\partial^2 f}{\partial \sigma \partial \sigma} = \frac{f_2}{\rho} \left( \mathbf{P} - \mathbf{n} \otimes \mathbf{n} \right) + \frac{f_3}{\rho^3} \left[ \frac{\sqrt{6}}{\rho} J_{3,0} - \xi (\mathbf{P} + \mathbf{n} \otimes \mathbf{n} - \mathbf{r} (\mathbf{j} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{j})) \right]
\]

\[+ \left[ \mathbf{n} \right] \otimes \begin{bmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{bmatrix} \mathbf{n} \]

5.1. Numerical examples

5.1.1. Constitutive problem. The proposed concept is applied to the numerical solution of plastic flow due to pure shear. The deviation of the return path from the radial direction is demonstrated best at stress states, where shear dominates.

The data of the material are given in terms of Young’s modulus \( E = 1 \times 10^4 \) N/mm², Poisson’s ratio \( \nu = 0.3 \), tensile yield stress \( \sigma_1 = 173.2 \) N/mm² and the shape factor \( \beta = \frac{1}{2} \) for the yield curve. The ‘loading’ is given in terms of the only non-zero strain components \( \varepsilon_{11} = -0.03 \) and \( \varepsilon_{22} = +0.03 \).

The non-zero components of the trial stress state are evaluated as \( \sigma_{11}^* = -230.8 \) N/mm² and \( \sigma_{22}^* = 230.8 \) N/mm². The constitutive algorithm computes the final stress state after five iterations to \( \{ \sigma_{11}, \sigma_{22}, \sigma_{23}, \sigma_{12} \} = \{-116.1, 0, 0\} \).

The convergence of the four unknowns during the Newton-Raphson iteration is given in Table I in terms of their increments, starting from their trial values.

5.1.2. Boundary value problem. The perforated tension strip under plane strain conditions is analysed, where use is made of the symmetry. The finite element mesh is taken as in Reference 8, p. 660. The material data are assumed to be \( E = 7000 \) N/mm², \( \nu = 0.2 \), \( c_4 = 24.3 \) N/mm² and the three different values for the shape factor of the yield curve are \( \beta = -\frac{1}{4}, 0, \frac{1}{4} \).

The load steps are chosen as prescribed displacement increments of 0.01 mm, imposed on the shorter edges of the rectangular structure (Figure 2). Starting with an initial displacement of
Table 1. Initial values and increments (in exponential representation) of unknowns

<table>
<thead>
<tr>
<th>Step</th>
<th>$\rho^*$, $\Delta \rho$</th>
<th>$\xi^*$, $\Delta \xi$</th>
<th>$\Lambda^*$, $\Delta \Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trial</td>
<td>326 36</td>
<td>0 00</td>
<td>0 00</td>
</tr>
<tr>
<td>1</td>
<td>- 0.162E + 03</td>
<td>- 0.559E + 00</td>
<td>0.162E + 03</td>
</tr>
<tr>
<td>2</td>
<td>- 0.132E + 02</td>
<td>0.489E - 01</td>
<td>0.198E + 02</td>
</tr>
<tr>
<td>3</td>
<td>0.169E + 00</td>
<td>- 0.408E - 02</td>
<td>- 0.352E + 00</td>
</tr>
<tr>
<td>4</td>
<td>0.145E - 03</td>
<td>- 0.260E - 05</td>
<td>- 0.364E - 03</td>
</tr>
<tr>
<td>5</td>
<td>0.132E - 09</td>
<td>- 0.404E - 11</td>
<td>- 0.282E - 09</td>
</tr>
</tbody>
</table>

![Figure 2. Rectangular structure with circular hole (distances in mm)](https://example.com/f2.png)

![Figure 3. Load–displacement diagrams](https://example.com/f3.png)

0.04 mm, the analysis is carried out in six consecutive steps up to an end displacement of 0.10 mm. The resulting force $R$, acting on the boundary of a quarter of the strip, is plotted in Figure 3 for each value of $\beta$. The two load–displacement curves for $\beta = -\frac{1}{3}$ and $+\frac{1}{3}$ tend to an ultimate load of 151 and 150 N, respectively. At the final elongation $u = 0.10$ mm the stress state for the weakest cross-section at the circular boundary approaches for $\beta = \frac{1}{3}$ an angle $\phi$ of about 45°, measured with respect to the $S_\tau$-axis in the deviatoric plane (see Figure 1), where $S_1$ is the principal component of the stress deviator $S$. For $\beta = -\frac{1}{3}$ the corresponding value of the angle between the position vector of the stress state and the $S_\tau$-axis is about 14° at the same location and
Table II. Incremental energy norm (in exponential representation)

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy norm</td>
<td>4.83E+00</td>
<td>1.05E-02</td>
<td>3.00E-05</td>
<td>3.11E-10</td>
<td>2.36E-19</td>
</tr>
</tbody>
</table>

 elongation of the structure. For both stress states the associated 'length of the stress deviator', measured by \( \| S \| = \rho \), exceeds the one for all stress states on the von Mises yield surface, for which \( \beta = 0 \).

The convergence behaviour is reflected by the energy norm in terms of the incremental displacements, listed in Table II for the load step from 0.06 to 0.07 mm in the case of \( \beta = 1/2 \). It is characteristic for all other load steps, when the Newton–Raphson method is employed.

6. CONCLUDING REMARKS

In this work an efficient method is presented for the numerical time integration of the governing equations in isotropic elastoplasticity. An estimate of the cost can be made for each iteration step of Newton’s method, based on the number of floating point operations. The total cost of Gaussian elimination, excluding the operations on the right-hand side and other computational overhead, is given for a dense, unsymmetric matrix of order \( n \) according to the formula

\[
\frac{2}{3} n^3 - \frac{1}{2} n^2 - \frac{1}{6} n.
\]

For perfect plasticity with seven unknowns, the comparison shows a ratio of 6:1 in favour of the proposed concept over the iterative solution with all components of the stress tensor as unknowns.

The specific algebraic structure of many yield functions offers an even more advantageous solution strategy. Additional hardening variables, in general, diminish the computational advantage. However, some practical hardening models allow partial elimination of the internal variables during the iterative solution process to regain the small set of coupled, non-linear equations.

The question of accuracy of the elastic predictor and plastic corrector methods still needs to be addressed for the class of general isotropic yield surfaces. In order to achieve improved numerical accuracy, any polynomial extrapolation method for the predictor \( \sigma^* \) may be used, allowing plastic flow to occur in the predictor step as well. Since the predictor \( \sigma^* \) usually violates the yield criterion \( F = 0 \), a corrector step may follow, such that the yield criterion is satisfied at the end of the time step and only plastic flow is assumed to occur in the correcting step. The proposed method is well suited for such correcting steps.

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