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DETERMINATION OF EFFECTIVE MATERIAL FUNCTIONS FOR LINEAR VISCOELASTIC FIBROUS COMPOSITES WITH MICRO-MECHANICAL MODELS

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Abstract. For the numerical computation of structures made of fibrous composites it is necessary to determine the overall behavior of the homogenized material replacing the composite. In many analysis the rate-dependent material behavior must be accounted for. A numerical procedure is presented first to identify the parameters of linear viscoelastic material models. By means of an analytic approximation, known as the micromechanical method of cells, the average elastic properties of the homogenized material are calculated from the material properties and volume fractions of the individual phases. The micromechanical cell model is applied in the Laplace domain for the determination of the effective properties of the rate-dependent composite with a viscoelastic matrix material. This methodology is based on the application of the correspondence principle of linear viscoelasticity. The numerical inversion into the time domain supplies the independent viscoelastic material functions of the homogenized composite material.

Key words: parameter identification, viscoelasticity, micromechanics, composites, homogenization

1 Introduction

An important objective of the theoretical description of composite material behavior is the prediction of the effective material properties by means of phenomenological continuum mechanics. The effective properties are taken for the macroscopic characterization, as the composite material is replaced by an equivalent homogeneous

continuum. The mathematical characterization of the overall effective properties of heterogeneous materials can take place in different ways. On the basis of experimental data of composite specimen it is possible to characterize the material behavior and predict appropriate material parameters. Micromechanical models represent another method for the approximation of the mechanical behavior of heterogeneous materials by a homogeneous continuum. The micromechanical approach serves as an aid for the modeling of the material behavior of composites on the basis of known material parameters and volume fractions of the constituents. In comparison to the experimental identification of composite materials the micromechanical model provides the material parameters for all fiber matrix combinations, volume percentages and geometrical arrangements with little effort. The fundamentals of the micromechanical approach, based on the representative unit cell, are outlined in the following.
Since the composite material shall consist of one or more polymeric phases, the time-dependency of mechanical behavior must be taken into account. The rate-dependency can be described by means of the theory of the linear viscoelasticity, which is represented briefly in the following. In addition, a numerical procedure is given for the parameter identification of the viscoelastic material functions.
The independent viscoelastic material functions for a fiber composite are determined with the micromechanical model presented. The viscoelastic correspondence principle serves as a basis for this method together with a numerical algorithm for the inversion of the material functions of the composite from the Laplace into the time domain.

2 Foundations of linear viscoelasticity

The phenomenon of the time-dependent mechanical behavior of materials can show up in different ways: stress relaxation under constant loading, decay of vibrations or strain and stress rate-dependence are some examples of viscoelastic material behavior. Thus, the constitutive equations, forming the connectivity between the stresses and the strains in the material, are functions of time. In the case of infinitesimal strains the material behavior of polymers can be described well with the theory of the linear viscoelasticity in many cases. Based on the Boltzmann superposition principle, the constitutive equations for linear viscoelastic materials follow:

$$\epsilon(t) = \int_{-\infty}^{t} J(t-\tau)\,\sigma'(\tau)d\tau \qquad\qquad \sigma(t) = \int_{-\infty}^{t} G(t-\tau)\,\epsilon'(\tau)d\tau, \tag{1}$$

where ϵ' and σ' are the time derivatives of the strain and stress history. The response of the material is a functional of the entire preceding strain or stress history. $J(t)$ and $G(t)$ are the viscoelastic material functions, describing the behavior of the material after the application of a constant stress or strain jump. $J(t) = \epsilon(t)/\sigma_0$ is the creep function and $G(t) = \sigma(t)/\epsilon_0$ the relaxation function. The rheological network of springs and dampers provides a sum of exponentials in time for these material functions, whose parameters must be determined from experiments .

2.1 Rheological models

By using the solution for the simple model of Kelvin and Voigt (parallel connection of spring and damper), the strain response for the generalized Kelvin–Voigt model (Fig. 1) can be determined from the superposition principle of the strains.

$$\epsilon(t) = \int_{-\infty}^{t} \underbrace{\left(\frac{1}{E_0} + \sum_{i=1}^{N} \frac{1}{E_i} \left(1 - \mathrm{e}^{-\frac{E_i}{\eta_i}(t-\tau)} \right) \right)}_{J(t-\tau)} \sigma'(\tau) d\tau \tag{2}$$

Eq. (2) represents the general constitutive relation of linear viscoelasticity of the „creep type", from which the creep function $J(t)$ can be read off. Similarly, the relaxation function for the generalized Maxwell model can be calculated by the fundamental model according to Maxwell (spring and damper in series) and the superposition of the stresses.

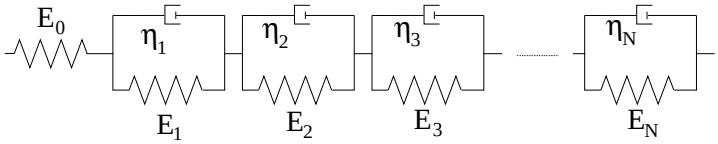

Fig. 1: Generalized Kelvin-Voigt-Model

2.2 Identification of the discrete viscoelastic spectrum

The material functions of the theory of linear viscoelasticity are finite Dirichlet Prony series. They are used as approximations for the experimental data as in the following: The generalized Kelvin-Voigt model provides the creep function

$$J(t) = J_0 + \sum_{i=1}^{i=N} J_i \left(1 - \mathrm{e}^{(-t/\tau_i)} \right) \tag{3}$$

with the discrete retardation spectrum $L(\tau)$: $\{J_i, \tau_i;\ i = 1, 2, \ldots N\}$; and the generalized Maxwell model the relaxation function

$$G(t) = G_0 + \sum_{i=1}^{i=N} G_i \, \mathrm{e}^{(-t/\tau_i)} \tag{4}$$

with the discrete relaxation spectrum $H(\tau)$: $\{G_i, \tau_i;\ i = 1, 2, \ldots N\}$. The unknown parameters of the functions (3) and (4) represent the discrete viscoelastic spectra and completely determine the time-dependent behavior. J_i resp. G_i are the factors of the kernel functions and define the spectrum strengths. The exponents of the kernel function describe the respondance times $\tau_i = E_i/\eta_i$ (retardation or relaxation times). In the following the parameters $g_i = G_i/(G_0 + \sum_{i=1}^{i=N} G_i)$ of the normalized relaxation spectrum are used. The identification of these parameters for the known test data requires the solution of a nonlinear constraint optimization task on the basis of the least square method.

Additional constraints must be postulated for the minimization problem in order to guarantee positive parameters for a physically realistic material behavior. However, in this report use is made of the similarity of the step function to the kernel functions, leading to a recursive algorithm, already proposed in [8] by Emri and Tschoegl for the estimation of the relaxation parameters. The algorithm enables the determination of the discrete spectrum on the basis of experimental data - either from tests with static or dynamic loads. In addition, it is possible to interconverte the viscoelastic material functions with this scheme. The procedure is explained at the reconstruction of a relaxation function with a known relaxation spectrum. The relaxation spectrum chosen consists of 4 normalized spectrum lines $g_i(t)$ (Fig. 2 a). The graphs of the different time-dependent contributions are shown in Fig. 2 b to 2 e on a logarithmic time scale. Fig. 2 f represents the graph of the complete relaxation function, resulting from the assumed spectrum.

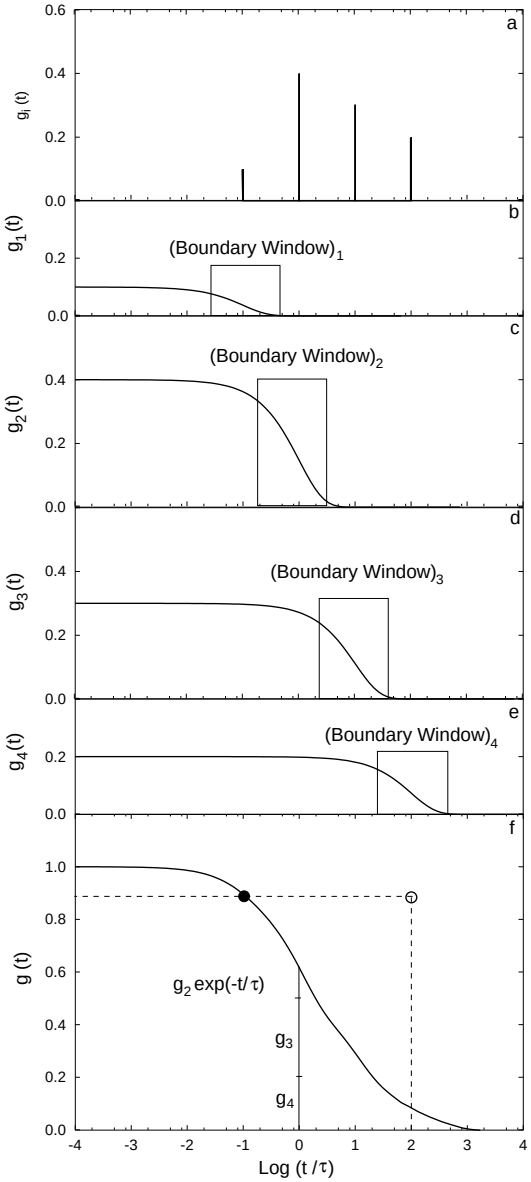

Fig. 2: Relaxation fct. $g_i(t) = g_i \; \mathrm{e}^{-t/\tau_i}$
$\tau_i = (0, 1, 1, 10, 100)$;
$g_i = (0.1, 0.4, 0.3, 0.2)$

It is possible to divide each function $g_i(t)$ into three parts: the „glassy" part, where the kernel function is virtually constant with a value equal to the strength g_i; the „transient" part, in which the graph of the curve is clearly time-dependent, and the „equilibrium" region, in which the graph of the curve almost vanishes. On the basis of this classification the value of the function for the sum of relaxation exponentials at a given time may be specified approximately in a simple way.

As an example the relaxation function is regarded in the vicinity $t_l \leq t \leq t_u$ at the point in time $t = \tau_2 = 1$ in detail (Fig. 2 f): The contributions of the exponentials $g_4(t)$ and $g_3(t)$ are approximately equal to the constants g_4 and g_3. However, the contribution of the exponential $g_1(t)$ vanishes. The contribution of $g_2(t)$ is time-dependent in this

area and cannot be simplified by a constant value in this range. Thus, the value of the relaxation function $g(t)$ in the interval $[t_l, t_u]$ is given in good approximation by the sum of the two nonzero constants and the time-dependent part $g_2(t)$:

$$g(t_l \le t \le t_u) = 0 + g_2\, \mathrm{e}^{-(t_l \le t \le t_u)/\tau_2} + g_3 + g_4. \tag{5}$$

This equation demonstrates the fact that those exponentials practically vanish, whose respondance times τ_i are much smaller than the timepoint t of the function value regarded ($\tau_i \ll t$), while the contributions of the "far" spectrum lines to the right of the vicinity ($\tau_i \gg t$) are actually constant in time. In the general case the following relationship holds approximately:

$$g(t_l \le t \le t_u) = 0 + g_k\, \mathrm{e}^{-(t_l \le t \le t_u)/\tau_k} + \sum_{i=k+1}^{i=N} g_i \tag{6}$$

For any datum point within the vicinity of the largest respondance time τ_N, the normalized strength is:

$$g(t_l \le t \le t_u) = g_N \mathrm{e}^{-(t_l \le t \le t_u)/\tau_N}. \tag{7}$$

With the help of the known respondance time τ_N and the function value of the exponential in the vicinity of τ_N the appropriate spectrum strength g_N can be determined directly from eq. (7). This may be taken as the starting relation for a recursive calculation of the discrete distribution of spectrum lines to be identified. For a selected set of suitable respondance times, beginning with the largest one, the associated spectrum strengths g_k are determined successively from the recursion equation

$$g_k = \frac{g(t_l \le t \le t_u) - \sum_{i=k+1}^{i=N} g_i}{\mathrm{e}^{-(t_l \le t \le t_u)/\tau_k}}. \tag{8}$$

This condition requires the selection of a single datum point in the vicinity of τ_k from the available set of test data. A practical definition of the term "vicinity" needs to be given.

The example shows that only certain test data may be chosen for the calculation of the appropriate spectrum strength. If, for example, the function value at time $t = \tau_1$ is used to determine the spectrum strength g_4 (marked by the filled circle in Fig. 2 f), then eq. (7) yields the false result

$$g_4 = g(t)|_{t=\tau_1}. \tag{9}$$

In Fig. (2 f) this result is marked by a partially filled circle. The optimal approximation of the chosen relaxation modulus would require now negative values for the spectrum line between τ_1 and τ_4 in order to compensate for the error. Obviously, the test data chosen for the estimation of the relaxation strength must be in accordance with the respondance time considered. Therefore, the terms *boundary* und *modeling window* are introduced. The *boundary window* is the intervall on the time axis, where the exponential function can describe the time-dependent response of the experimental data.

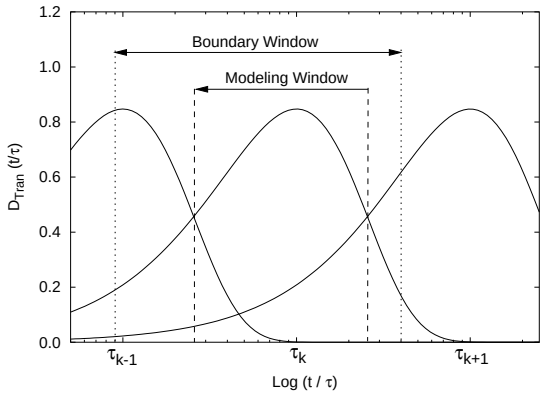

Fig. 3: Definition of the modeling window

In this area the first derivative of the kernel is clearly greater than zero. The *modeling window* comprises a closer subinterval of the *boundary window* - see fig. 3. This section is defined between the intersections of the curves for the first logarithmic derivative D_{Tran} of the chosen kernel functions in the next neighborhood.

Algorithm For the general case with an arbitrary number of data points the procedure is as follows: The absolute error Δ_j between the data points and the relaxation function is introduced and the least square norm E_k is formed with M function values $\hat{g}(t_j)$ at t_j.

$$E_k = \sum_{j=1}^{M} \left[g_0 + \sum_{i=m}^{k-1} g_i \mathrm{e}^{-t_j/\tau_i} + g_k \mathrm{e}^{-t_j/\tau_k} + \sum_{i=k+1}^{N} g_i \mathrm{e}^{-t_j/\tau_i} - \hat{g}(t_j) \right]^2 \tag{10}$$

The parameter m depends on the number of selected respondance times per decade. The minimization of the error function supplies the condition in eq. (11) for the determination of the spectrum strength g_k with the appropriate respondance time τ_k.

$$\frac{\partial E_k}{\partial \hat{g}_k} = 0 \longrightarrow g_k = \frac{\sum\limits_{j=1}^{M} \left[\hat{g}(t_j) - g_0 - \sum\limits_{i=m}^{k-1} g_i \mathrm{e}^{-t_j/\tau_i} - \sum\limits_{i=k+1}^{N} g_i \mathrm{e}^{-t_j/\tau_i} \right] \mathrm{e}^{-t_j/\tau_k}}{\sum\limits_{j=1}^{M} \mathrm{e}^{-2t_j/\tau_k}} \tag{11}$$

This equation is recursively applied, beginning with the largest respondance time τ_N. In the first cycle all spectrum strengths g_k are set to zero. In the following cycles any positive strength values will be saved and negative values are set to zero. The procedure is terminated as soon as the norm of changes of the parameters is smaller than a given criterion. For the application to non smoothed test data the procedure is modified by introducing the relative error $\delta_j = \frac{\Delta_j}{g(t_j)}$.

2.3 Viscoelastic correspondence principle

Viscoelasticity problems can be reduced to elastic ones by the application of the Laplace transform, an integral transform method with a linear operator. It serves as

a solution procedure for linear differential equations with constant coefficients and is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st} dt \tag{12}$$

The original function is $f(t)$, defined in the time domain, whereas the function $F(s)$ is called the Laplace function, existing in the Laplace or image domain. The variable s defines the complex Laplace variable $s = \sigma + i\omega$, where σ should not be confused with the stress. By means of the derivative theorem of the Laplace theory

$$\mathcal{L}\{f'(t)\} = s\, F(s) - f(0), \tag{13}$$

and the convolution theorem

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \mathcal{L}\left\{ \int_0^t f_1(u)\, f_2(t-u) du \right\} = F_1(s) F_2(s) \tag{14}$$

the constitutive equations of linear viscoelaticity (1) can be transformed into the image domain:

$$\sigma(t) = \int_{-\infty}^t G(t-\tau)\, \epsilon'(\tau) d\tau \quad \Rightarrow \quad \bar{\sigma}(s) = s\, \bar{G}(s)\, \bar{\epsilon}(s) \tag{15}$$

$$\epsilon(t) = \int_{-\infty}^t J(t-\tau)\, \sigma'(\tau) d\tau \quad \Rightarrow \quad \bar{\epsilon}(s) = s\, \bar{J}(s)\, \bar{\sigma}(s). \tag{16}$$

Eqs. (15) and (16) show the analogy of viscoelasticity with linear elasticity, if the transformed viscoelastic characteristic material functions in the image domain are interpreted as the elastic material parameters:

$$s\, \bar{G}(s) \Longleftrightarrow E \qquad ; \qquad s\, \bar{J}(s) \Longleftrightarrow S. \tag{17}$$

The correspondence principle permits the determination of the viscoelastic material functions from the associated solution for elastic material behavior in the Laplace domain. The micromechanical analysis of viscoelastic composites consists of three steps, if use is made of the Laplace transform:

I. Laplace transform of the material functions: The material equations of the components in the time domain must be transferred by means of the Laplace transform into algebraic equations in the image domain. The transformed material functions (3) and (4) become:

$$\bar{J}(s) = \frac{J_0}{s} + \sum_{i=1}^N \frac{J_i}{\tau_i s (s + 1/\tau_i)} \tag{18}$$

$$\bar{G}(s) = \frac{G_0}{s} + \sum_{i=1}^N \frac{G_i}{(s + 1/\tau_i)} \tag{19}$$

II. Homogenization in the Laplace domain: By applying the correspondence principle, the elastic material parameters are replaced by the s-multiple of the Laplace transformed material functions in accordance with eq. (17).

$$s\bar{J}(s) = J_0 + \sum_{i=1}^{N} \frac{J_i}{\tau_i(s+1/\tau_i)} \tag{20}$$

$$s\bar{G}(s) = G_0 + \sum_{i=1}^{N} \frac{G_i s}{(s+1/\tau_i)} \tag{21}$$

By means of the micromechanical model, see chapter 3, the material equations of the composite in the Laplace domain are determined.

III. Inverse Laplace transform: The inverse integral transform from the image into the time domain provides the material functions of the composite.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \tag{22}$$

The operator $\mathcal{L}^{-1}$ is the inverse operator to $\mathcal{L}$, defined by:

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t). \tag{23}$$

The explicit determination of the inverse operator is a fundamental problem of the theory of integral transforms. Generally, the following ways are available for the inverse transformation into the time domain: Use of tables with corresponding original and Laplace functions, direct computation of the inversion integral if possible or series expansion. However, only in a few cases the inverse Laplace transform can be performed analytically. Usually numerical procedures are necessary such as the use of a Fourier series expansion or a Laguerre polynomial expansion. No procedure is well suited for all function inversions. Pindera [9] used the method of Bellman [3], which gave a good approximation of the effective material functions for small values of time t. The numerical approach of D'Amore et al. [6] is used here and will be briefly presented. It is based on a Fourier series expansion. The inverse transform of Riemann

$$f(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{ts} F(s) ds, \quad s = \sigma + i\omega, \quad \sigma > \sigma_0, \quad t > 0 \tag{24}$$

represents an integral with a complex path of integration, where σ_0 is the abscissa of convergence of F in the right half plane. According to De Hoeg et. al. [7] this integral can be calculated by the trapezoidal rule with an incrementation of $h = \pi/T$. Hence, it follows:

$$\tilde{f}_\infty(t) = \frac{\mathrm{e}^{\sigma t}}{T} Re\left[\frac{F(\sigma)}{2} + \sum_{k=1}^{\infty} F(\sigma + \frac{ik\pi}{T}) \mathrm{e}^{\frac{ik\pi t}{t}}\right] \tag{25}$$

with the truncation error $\eta(t) = f(t) - \tilde{f}_\infty(t)$. For N terms of the Fourier series the approximation of $\tilde{f}_N(t)$ with

$$\tilde{f}_N(t) = \frac{\mathrm{e}^{\sigma t}}{T} Re\left[\frac{F(\sigma)}{2} + \sum_{k=1}^{N} F(\sigma + \frac{ik\pi}{T})\mathrm{e}^{\frac{ik\pi t}{t}}\right] \tag{26}$$

provides the discrete original function at given points in time t for an analytical function in the Laplace domain.

3 Micromechanics of inhomogeneous media

The micromechanical modeling of inhomogeneous media aims at the determination of the effective (average) material properties for the equivalent homogeneous comparison material by means of the mechanical properties of the components and their geometrical arrangement in the composite. The description of the material behavior is based on the representative volume element (RVE), whose structure should be typical for the composite, small in the comparison to the macrostructure and large in the comparison to the microstructure. The effective material properties are defined between the averaged stresses and the averaged strains of the composite.

$$\overline{\boldsymbol{\sigma}} = \mathbf{C}^* \, \overline{\boldsymbol{\epsilon}} \quad \text{resp.} \quad \overline{\boldsymbol{\epsilon}} = \mathbf{S}^* \, \overline{\boldsymbol{\sigma}} \tag{27}$$

$$\overline{\boldsymbol{\sigma}} = \frac{1}{V} \int_V \boldsymbol{\sigma}(x) dV \quad \text{and} \quad \overline{\boldsymbol{\epsilon}} = \frac{1}{V} \int_V \boldsymbol{\epsilon}(x) dV \tag{28}$$

Since the averaged stress and strain values may only be determined from the exact solution of the stress and strain fields - see eq. (28) - simplifying assumptions must be made for the micromechanical models in order to find an approximation for the material parameters of the composite.

3.1 Concentration tensors of Hill

The fourth order concentration tensors $\bar{\mathbf{A}}^{(i)}$ or $\bar{\mathbf{B}}^{(i)}$ of Hill are introduced for the determination of the material properties. The average composite stress $\overline{\boldsymbol{\sigma}}$ is expressed as a function of the average phase strain $\boldsymbol{\epsilon}^{(i)}$, the stiffness tensors $\mathbf{C}^{(i)}$ of the phases and their volume fractions v_i.

$$\overline{\boldsymbol{\sigma}} = v_1 \mathbf{C}^{(1)} \overline{\boldsymbol{\epsilon}}^{(1)} + v_2 \mathbf{C}^{(2)} \overline{\boldsymbol{\epsilon}}^{(2)} \tag{29}$$

With the help of of the phase averaged concentration tensors $\bar{\mathbf{A}}^{(i)}$ and $\bar{\mathbf{B}}^{(i)}$ a relationship is set up between the averaged stresses $\overline{\boldsymbol{\sigma}}^{(i)}$ in the phases and the averaged ones of the composite, as well as between the average strains of the phases and those of the composite.

$$\overline{\boldsymbol{\sigma}}^{(1)} = \bar{\mathbf{B}}^{(1)} \overline{\boldsymbol{\sigma}} \quad ; \quad \overline{\boldsymbol{\sigma}}^{(2)} = \bar{\mathbf{B}}^{(2)} \overline{\boldsymbol{\sigma}} \tag{30}$$

$$\overline{\boldsymbol{\epsilon}}^{(1)} = \bar{\mathbf{A}}^{(1)} \overline{\boldsymbol{\epsilon}} \quad ; \quad \overline{\boldsymbol{\epsilon}}^{(2)} = \bar{\mathbf{A}}^{(2)} \overline{\boldsymbol{\epsilon}} \tag{31}$$

With these definitions the average stress of the composite may be given in terms of the average composite strains $\overline{\boldsymbol{\epsilon}}$:

$$\overline{\boldsymbol{\sigma}} = v_1 \mathbf{C}^{(1)} \bar{\mathbf{A}}^{(1)} \overline{\boldsymbol{\epsilon}} + v_2 \mathbf{C}^{(2)} \bar{\mathbf{A}}^{(2)} \overline{\boldsymbol{\epsilon}} \tag{32}$$

The effective stiffness of the composite becomes:

$$\mathbf{C}^* = v_1 \mathbf{C}^{(1)} \bar{\mathbf{A}}^{(1)} + v_2 \mathbf{C}^{(2)} \bar{\mathbf{A}}^{(2)} \tag{33}$$

and with the identity $\mathbf{I} = v_1 \bar{\mathbf{A}}^{(1)} + v_2 \bar{\mathbf{A}}^{(2)}$ it follows:

$$\mathbf{C}^* = \mathbf{C}^{(2)} + v_1 (\mathbf{C}^{(1)} - \mathbf{C}^{(2)}) \; \bar{\mathbf{A}}^{(1)}. \tag{34}$$

Similarly, the effective compliance tensor can be formed.

$$\mathbf{S}^* = v_1 \mathbf{S}^{(1)} \bar{\mathbf{B}}^{(1)} + v_2 \mathbf{S}^{(2)} \bar{\mathbf{B}}^{(2)} \tag{35}$$

3.2 Elementary rules of mixture

The simplest micromechanical models represent the rules of mixture according to Voigt and Reuss. Voigt made the assumption that the strains in the phases $\boldsymbol{\epsilon}^{(i)}$ are equal to the strains of the composite material. Hence, the concentration tensors $\bar{\mathbf{A}}^{(1)}$ and $\bar{\mathbf{A}}^{(2)}$ become the unit tensors $\mathbf{I}$. The effective stiffness tensor results in:

$$\mathbf{C}^* = v_1 \, \mathbf{C}^{(1)} + v_2 \, \mathbf{C}^{(2)}. \tag{36}$$

Reuss assumed that the phases of the composite material are subjected to uniform stress $\overline{\boldsymbol{\sigma}}$, corresponding to the average stress. Therefore, the concentration tensors $\bar{\mathbf{B}}^{(1)}$ and $\bar{\mathbf{B}}^{(2)}$ equal the unit tensor $\mathbf{I}$. The effective compliance tensor is:

$$\mathbf{S}^* = v_1 \, \mathbf{S}^{(1)} + v_2 \, \mathbf{S}^{(2)}. \tag{37}$$

Both models represent rough approximations, which, however, form upper and lower bounds in the case of the Voigt respectively the Reuss approach.

3.3 Method of cells

The method of cells, proposed by Aboudi, is a micromechanical model for the determination of the material properties of fiber-reinforced composites. The composite material shall consist of continuous square fibers in the x_1-direction, which are arranged double periodically in the $x_2 - x_3$-plane (Fig.4 a). This supposition allows to identify a RVE, consisting of four subcells (Fig. 4 b). A bilinear displacement assumption

$$u_i^{(\beta\gamma)} = w_i^{(\beta\gamma)} + \bar{x}_2^{(\beta)} \phi^{(\beta\gamma)} + \bar{x}_3^{(\gamma)} \psi^{(\beta\gamma)} \tag{38}$$

is specified in the local coordinate systems of each subcell. The material behavior of the cells shall be either isotropic or transversely isotropic. Continuity of the displacements is required at the interfaces between the subcells of the RVEs as well as at the boundaries between the RVEs, leading to a total of 13 independent equations:

$$[A_G]\{\bar{\epsilon}_s\} = [J]\{\bar{\epsilon}\} \tag{39}$$

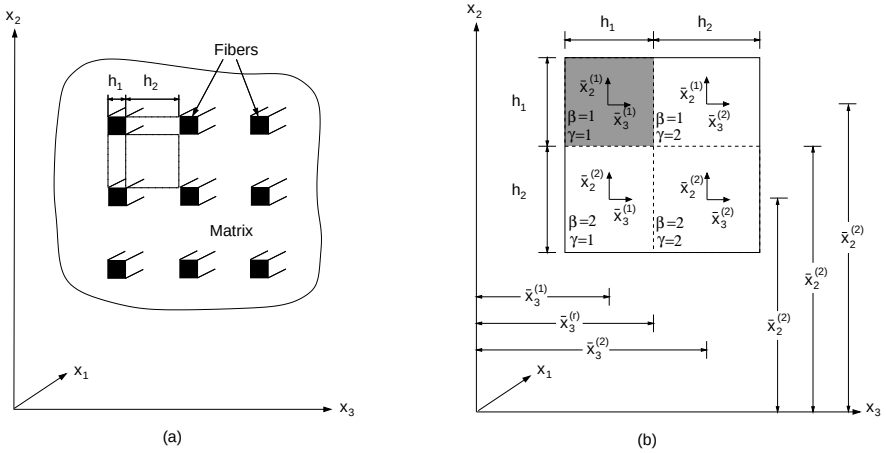

Fig. 4: (a) Composite with double periodic arrays of fiber reinforcement in x_1-direction (b) Representative cell with four subcells $\beta, \gamma = 1, 2$

The coefficients of the matrices $[A_G]$ and $[J]$ result from the geometry of the RVEs. $\{\bar{\epsilon}_s\}$ is the column vector of the 24 unknown strain components in the four subcells; $\{\bar{\epsilon}\}$ represents the averaged strain components.
The continuity condition of the stress components in the interface of the subcells of the RVE form another 11 independent equations:

$$[A_M]\{\epsilon_s\} = 0 \tag{40}$$

The matrix $[A_M]$ contains only known coefficients, depending on the material parameters of the constituents. The last two equations are solved simultaneously for the strain components of the subcells. The matrix $[A]$ comprises the coefficients of the four concentration tensors $\bar{\mathbf{A}}^{(i)}$ for the subcells.

$$\{\bar{\epsilon}_s\} = [A]\{\bar{\epsilon}\} \qquad \text{mit} \qquad [A] = \begin{bmatrix} A_G \\ A_M \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ J \end{bmatrix} \tag{41}$$

The effective stiffness tensor of the composite with orthotropic symmetry can be calculated by means of the concentration tensors $\bar{\mathbf{A}}^{(i)}$. The rotational average about the x_1-axis of the orthotropic stiffness tensor yields the material parameters for transverse isotropy [1].

4 Numerical simulation

In the following example the viscoelastic behavior of a composite is predicted, which consists of a viscoelastic matrix reinforced by elastic fibers. The relationship between the volume fractions of the phases and the time-dependent response is generally nonlinear. Therefore, simple rules of mixture are not applicable. The elasticity moduli of the isotropic linear elastic fiber reinforcement are $E_f = 85.5$ GPa and $\nu_f = 0.2$. The isotropic linear viscoelastic matrix has a time-independent bulk modulus of $K(t) = 5.47$ GPa and a time-dependent relaxation function for the shear modulus of $G(t) = 1.47 + 0.22\ \mathrm{e}^{-0.15t}$ [GPa].

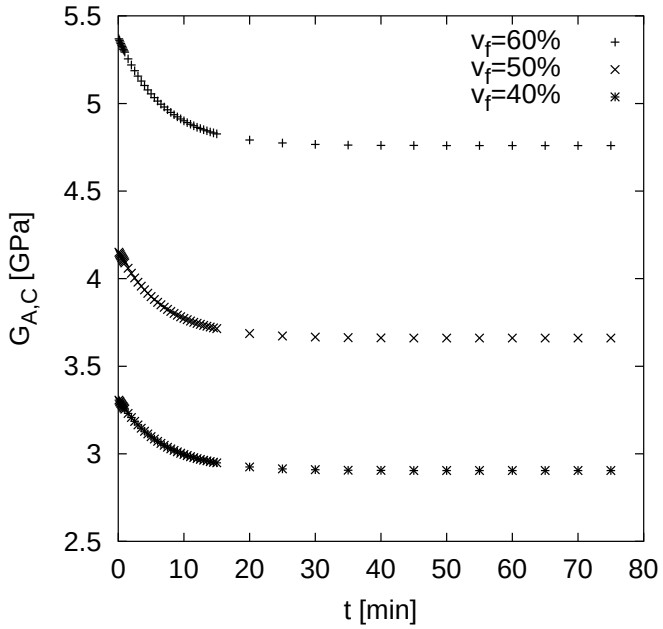

Fig. 5: Predicted axial shear relaxation function G_A^* of the composite

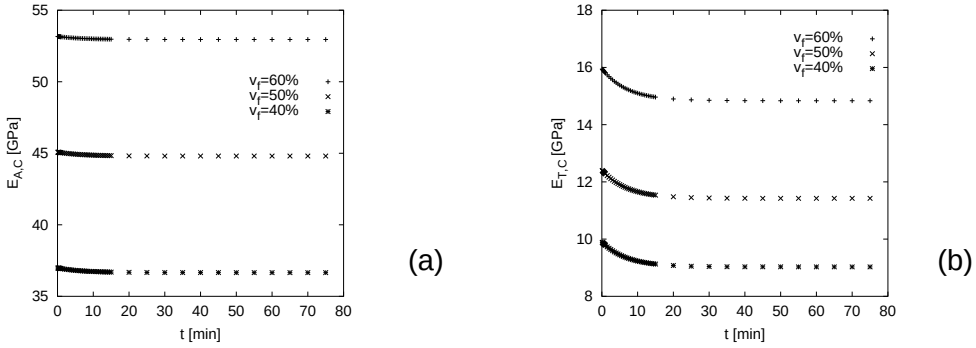

Fig. 6: Predicted relaxation function of (a) the axial elasticity modulus E_A^* and (b) the transversal elasticity modulus E_T^* of the composite

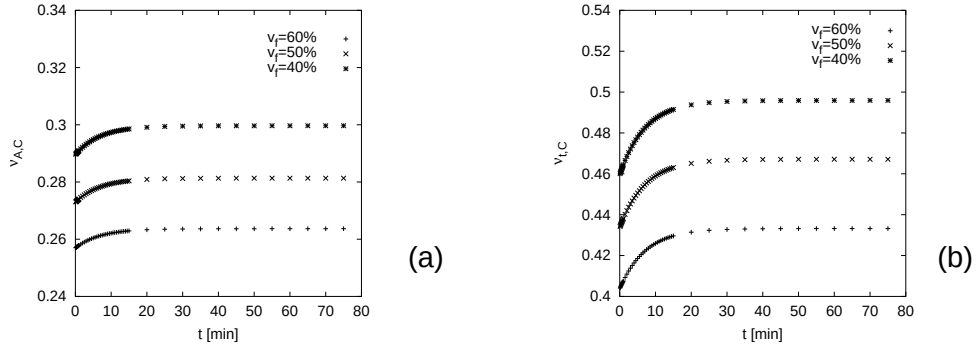

Fig. 7: Time-dependence of (a) axial Possions ratio ν_A^* and (b) transversal Possions ratio ν_t^* of the composite

Figs. 5,6 and 7 show the graphs for the five independent material functions of the transverse isotropic material behavior for different fiber volume fractions. With the help of the identification algorithm presented, the spectra may be determined for the five independent material functions of the composite. In fig. 8 (a) the discrete relaxation function G_A^* for a fiber volume content of 50% is approximated by a Dirichlet Prony series with the relaxation spectrum given in fig 8 (b). Fig. 9 shows the convergence of the algorithm, since the error tolerance tends towards zero during the iteration. The iterative scheme is stable, since only small changes of the difference norm between successive iterates are encountered.

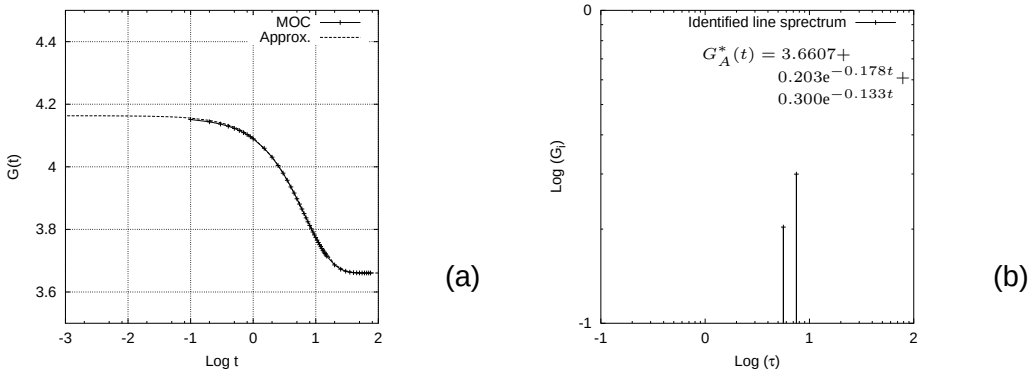

Fig. 8: (a) Approximation of the discrete relaxation function G_A^* from micromechanics
(b) Spectrum of the relaxation function G_A^*

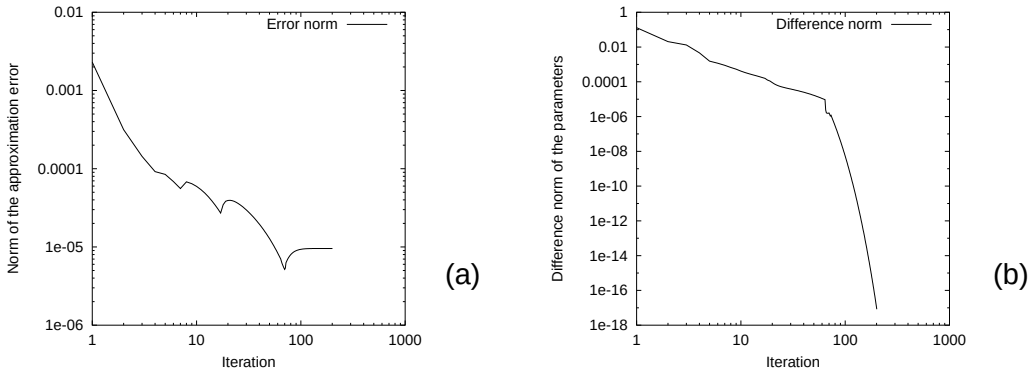

Fig. 9: (a) Approximation error during iteration
(b) Difference norm of the parameters between successive iterates

5 Summary

In this article the micromechanical method of cells was presented for the determination of the effective elastic and viscoelastic moduli of composite materials. The viscoelastic correspondence principle as well as a numerical procedure for the inversion of the relaxation function in the Laplace domain were outlined. The spectrum of the exponential series, approximating the discrete relaxation function of the composite, is found by the numerical procedure of Emri and Tschoegl. The similarity of the

kernel functions with a step function on the logarithmic time scale serves as basis for the development of the algorithm. The appropriate spectrum strengths are determined from a selected set of suitable respondance times. This approach leads to a recursive algorithm, where only a decoupled set of linear equations must be solved in each recursion cycle. No inversion of a possibly ill-conditioned matrix is needed and, thus, the procedure is stable, i.e. small disturbances of the experimental data cause only small changes of the result.

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