Micromechanical modeling of viscoelastic composites with compliant fiber–matrix bonding

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Abstract

The micromechanical method of cells is used to calculate the average time-dependent constitutive properties of the homogenized substitute continuum from the viscoelastic material properties and volume fractions of the individual phases as well as from the interphase behavior of polymeric fiber composites. Two approaches are investigated to establish and solve the micromechanical model equations of the rate-dependent material phases. The first one is based on the LAPLACE-transformation of the time-dependent material functions and the application of the correspondence principle of linear viscoelasticity to the governing equations of the micromechanical model. In the second approach a numerical time integration scheme is developed to compute the convolution integrals.

For the study of the influence of compliant fiber–matrix bonds alternative models are proposed: A representative volume element with three constituents accounts for the properties of the flexible interphase between the fibers and the matrix. The three-phase model is then compared to a two-phase model with a time-dependent flexible interface to simulate the compliant fiber–matrix bond. Various examples show the performance of both models and the numerical algorithms presented.

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1. Introduction

The stress analysis of polymeric fiber composite structures with commercial finite element codes requires the effective macroscopic material functions of the homogenized continuum, which replaces the heterogeneous composite in the numerical solution of the boundary value problem. The rate-dependent material behavior and the mechanical properties of the interphase between the reinforcing fibers and the matrix must be frequently accounted for in structural analysis of composite materials. Therefore, numerous experimental investigations in composite research focus on the characterization of the influence of the interphase and the fiber–matrix bond with respect to the mechanical performance of composite materials e.g. Wacker [1] and Kim [2]. The experimental results motivate the development of analytical micromechanical models with...
the interphase region as a third material component. In this paper the micromechanical method of cells, known as an analytical homogenization scheme, is used to study two different bonding models: A flexible interphase region with a finite thickness and an imperfect interface with a compliant bonding. The interphase has its own material properties, different from those of the matrix or the fibers, due to coatings on the fibers or chemical treatment of the fiber surface. The compliant interface bonding results from the manufacturing process, causing voids or other localized mechanical imperfections.

The time-dependent nature of glass–fiber reinforced polymers is described by the linear theory of viscoelasticity. Model equations for a rate-dependent interface between the viscoelastic material constituents are proposed in terms of the displacement discontinuity as a functional of the history of the tractions across the interface.

Imperfect bonding between the material phases has been studied previously: e.g. Benveniste [3], Gosz et al. [4], Brinson and Knauss [5], Aboudi [6], Gerlach [7]. Viscoelastic interface models have been included into the generalized self-consistent scheme or the composite cylinder assemblage. In this paper we consider a viscoelastic interface between the two viscoelastic components of fibers and matrix.

The compliant time-dependent material functions of the interface are incorporated into the continuity equations of the generalized method of cells (GMC). The outcoming convolution integrals in the viscoelastic model equations are solved by either the LAPLACE transformation together with the correspondence principle of the linear theory of viscoelasticity or by numerical time integration. In the first case the numerical inversion of the LAPLACE transformed material functions of the composite into the time domain supplies the macroscopic viscoelastic response functions of the homogenized continuum. In the second approach the time integrals are solved by a time-stepping scheme. It yields the stress or the strain response in time due to any defined boundary and loading conditions of the representative volume element (RVE). Both results are compared to each other.

2. Micromechanics for elastic materials

The micromechanical modeling of inhomogeneous media aims at the determination of the effective (average) material properties for the equivalent homogeneous comparison material by means of the mechanical properties of the components and their geometrical arrangement in the composite. The description of the linear material behavior is based on the RVE, whose structure should be typical for the composite, small in comparison to the macrostructure and large in comparison to the microstructure. The effective material properties are defined between the volume averaged stresses $\langle \sigma \rangle$ and the averaged strains $\langle \varepsilon \rangle$ of the composite with the elastic material stiffness tensor $C^*$ and the compliance $S^*$ of fourth order.

$$\langle \sigma \rangle = \mathbb{C}^* : \langle \varepsilon \rangle \quad \text{and} \quad \langle \varepsilon \rangle = \mathbb{S}^* : \langle \sigma \rangle,$$

(1)

where the colon : stands for a double contraction and the parenthesis $\langle \rangle$ for the volume average according to

$$\langle \sigma \rangle = \frac{1}{V} \int_V \sigma(x) \, dV \quad \text{and} \quad \langle \varepsilon \rangle = \frac{1}{V} \int_V \varepsilon(x) \, dV$$

(2)

with $x$ as the position vector. Since the averaged stress and strain values may only be determined from the exact solution of the stress and strain fields—see Eq. (2), simplifying assumptions must be made for the micromechanical models in order to find an approximation for the constitutive parameters of the composite. The usual micromechanical approach introduces the fourth order concentration tensors according to HILL. With the help of the phased-averaged strain concentration tensor $A^{(i)}$ of fourth order a mathematical relationship is set up between the average strain $\langle \varepsilon^{(i)} \rangle$ in the individual phases $(i)$ and the averaged strain $\langle \varepsilon \rangle$ of the composite.

$$\langle \varepsilon^{(i)} \rangle = A^{(i)} : \langle \varepsilon \rangle.$$  

(3)

With the definitions above the average stress of the elastic composite may be given in terms of the average composite strains. Hence, the effective stiffness of the composite $C^*$ can be read off in terms of the concentration tensors $A^{(i)}$ and the stiffness tensors $C^{(i)}$ of the components.
\[
(\sigma) = \sum_{i=1}^{N} c_i C^{(i)} : A^{(i)} : (\epsilon) \rightarrow C_s = \sum_{i=1}^{N} c_i C^{(i)} : A^{(i)},
\]

where \(c_i\) represents the volume fractions and \(N\) the number of phases.

The GMC, proposed in Paley and Aboudi [8] is used to determine the concentration tensors—see Herakovich [9]. The cells model has proven to be very efficient in representing the elastic and inelastic behavior of fiber-reinforced unidirectional composite materials—see Aboudi [10]. In the two-dimensional formulation of this analytical model the composite shall consist of continuous fibers in the \(x_1\)-direction, which are arranged double periodically in the \(x_2-x_3\)-plane (Fig. 1(a)). Due to this assumption it is possible to identify a unit cell as a RVE, which is divided into an arbitrary number of subdomains, called subcells (Fig. 1(b)) with dimensions \(h_\beta\) and \(l_{\gamma}\). For each subcell, subscripted with the indices \((\beta\gamma)\), a bilinear displacement field is assumed and formulated in the local coordinate systems. The material behavior of the cells shall be either isotropic or transversely isotropic. Continuity of the displacements \(u_i\) in the local coordinates \(\bar{x}_j\) is required at the interfaces between the subcells of the RVEs as well as at the boundaries between adjacent RVEs. The local continuity conditions

\[
u_i^{(\beta\gamma)} \big|_{\bar{x}_2=h_\beta} = u_i^{(\beta\gamma)} \big|_{\bar{x}_2=0} \quad \gamma = 1, 2, 3, \ldots, N_{\gamma},
\]

are relaxed and shall be satisfied on average only along the plane \(\bar{x}_3 = \text{const}\).

\[
\int_{-\frac{h_\beta}{2}}^{\frac{h_\beta}{2}} u_i^{(\beta\gamma)} \left|_{\bar{x}_2=\frac{h_\beta}{2}} \right| \, \text{d}x_2 = \int_{-\frac{h_\beta}{2}}^{\frac{h_\beta}{2}} u_i^{(\beta\gamma)} \left|_{\bar{x}_2=-\frac{h_\beta}{2}} \right| \, \text{d}x_2, \quad \gamma = 1, 2,
\]

The same procedure is carried out for the second continuity equation along the interface \(\bar{x}_3 = \text{const}\).

The averaged compatibility equations can be written in matrix form in terms of the phase averaged strains \(\langle \epsilon_i \rangle\) and the effective strains \(\langle \epsilon \rangle\) of the composite—see Herakovich [9] or Paley et al. [8] for further details.

\[
A_G \langle \epsilon_i \rangle = J \langle \epsilon \rangle.
\]

The coefficients of the matrices \(A_G\) and \(J\) result from the geometry of the subcells in the RVE. \(\langle \epsilon_i \rangle\) is the column vector of the averaged unknown strain components \(\langle \epsilon^{(\beta\gamma)} \rangle\) in the \(N_{\beta}N_{\gamma}\)-subcells:

\[
\langle \epsilon_i \rangle^T := (\langle \epsilon^{(11)} \rangle, \langle \epsilon^{(12)} \rangle, \ldots, \langle \epsilon^{(N_{\beta}N_{\gamma})} \rangle)^T.
\]

The averaged strain tensor \(\langle \epsilon^{(\beta\gamma)} \rangle\) in subcell \((\beta\gamma)\) consists of the volume averaged strain components \(\langle \epsilon_{ij}^{(\beta\gamma)} \rangle\) in phase \(i\) filling up the cell.

\[
\langle \epsilon^{(\beta\gamma)} \rangle^T := (\langle \epsilon_{11}^{(\beta\gamma)} \rangle, \langle \epsilon_{22}^{(\beta\gamma)} \rangle, \langle \epsilon_{33}^{(\beta\gamma)} \rangle, \langle \epsilon_{12}^{(\beta\gamma)} \rangle, \langle \epsilon_{13}^{(\beta\gamma)} \rangle, \langle \epsilon_{23}^{(\beta\gamma)} \rangle)^T.
\]

Fig. 1. (a) Composite with double periodic arrays of fibers in \(x_1\)-direction, (b) Representative unit cell with subcells and nomenclature.
\( \langle \epsilon \rangle \) represents the averaged strain components in vector notation of the homogenized comparison material according to Eq. (2).

The equilibrium conditions for the phase averaged stresses \( \langle \sigma(\beta) \rangle \) at the interfaces of the subcells in the RVE are given by:

\[
\langle \sigma_{ij}^{(\beta)} \rangle = \langle \sigma_{ij}^{(\gamma)} \rangle ,
\]

\[
\langle \sigma_{j3}^{(\beta)} \rangle = \langle \sigma_{j3}^{(\gamma)} \rangle
\]

with \( j = 1, 2, 3; \beta = 1, \ldots, N_\beta \) and \( \gamma = 1, \ldots, N_\gamma, N_\beta \) and \( N_\gamma \) denote the number of subcells in each direction and \( \beta \) and \( \gamma \) are defined as:

\[
\hat{\beta} = \begin{cases} 
\beta + 1, & \beta < N_\beta \\
1, & \beta = N_\beta 
\end{cases}
\]

\[
\hat{\gamma} = \begin{cases} 
\gamma + 1, & \gamma < N_\gamma \\
1, & \gamma = N_\gamma 
\end{cases}
\]

The equilibrium conditions are linear equations and can be written in matrix form according to Herakovich [9] or Paley et al. [8]:

\[ \mathbf{A}_M \langle \epsilon_s \rangle = 0. \]

The matrix \( \mathbf{A}_M \) contains only known coefficients, depending on the material parameters of the constituents. Eqs. (8) and (14) may be combined in the following relationship in vector notation between the strains in all subcells \( \langle \epsilon_s \rangle \) and the macro strains in the composite \( \langle \epsilon \rangle \).

\[ \langle \epsilon_s \rangle = \mathbf{A} \langle \epsilon \rangle \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_G \\ \mathbf{A}_M \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{J} \\ 0 \end{bmatrix}. \]

The matrix \( \mathbf{A} \) comprises the elements of the fourth order concentration tensors \( \mathbf{A}_\beta^{(\beta)} \) for all subcells written in matrix form \( \mathbf{A}^{(\beta)} \). The effective stiffness matrix of the composite \( \mathbf{C}^* \) is finally calculated from the matrices \( \mathbf{A}^{(\beta)} \) of the concentration tensors.

\[ \mathbf{C}^* = \frac{1}{\lambda} \sum_{\beta=1}^{N_\beta} \sum_{\gamma=1}^{N_\gamma} h_{\beta\gamma} l_{\beta\gamma} \mathbf{C}^{(\beta)} \mathbf{A}^{(\beta)}. \]

3. Viscoelasticity

Based on the Boltzmann superposition principle, the constitutive equations for the theory of linear viscoelasticity may be written in the one-dimensional case as follows:

\[ \sigma(t) = \int_{-\infty}^{t} G(t - \tau) \frac{d\epsilon(\tau)}{d\tau} \, d\tau, \]

where \( \frac{d\epsilon(\tau)}{d\tau} \) is the time derivative of the strain history \( \epsilon(t) \). The kernel \( G(t) \) is the relaxation function, describing the material response due to a constant strain jump. On the basis of the generalized Maxwell model the relaxation function can be approximated by a finite Dirichlet–Prony series of the form

\[ G(t) = G_\infty + \sum_{i=1}^{N} G_i e^{-t/\tau_i} \]

with the discrete relaxation spectrum \( H(\tau_i) : \{ G_i, \tau_i; i = 1, 2, \ldots, N \} \), which completely determines the time-dependent behavior. The parameters \( G_i \) define the spectrum strengths in the kernel function. The coefficients in the argument of the exponentials in the kernel function are the relaxation times \( \tau_i \), which must be determined from the experimental data of relaxation tests together with the stiffness parameters \( G_i \).

3.1. Viscoelastic correspondence principle

Linear viscoelastic problems can be formally reduced to elastic ones by using the Laplace transform. Hence, the constitutive equation (17) becomes in the Laplace image domain:

\[ \tilde{\sigma}(s) = s \tilde{G}(s) \hat{\epsilon}(s), \]

which shows the analogy between viscoelasticity and linear elasticity, if the transformed viscoelastic material function \( \tilde{G}(s) \) in the image domain is interpreted as an elastic material parameter \( E \):

\[ s \tilde{G}(s) \equiv E. \]

The correspondence principle permits the calculation of the viscoelastic material functions from an associated elastic model by formally replacing the elasticity parameter \( E \) by the \( s \)-multiplied Laplace-transform \( s \tilde{G}(s) \) of the time-dependent constitutive functions \( G(t) \). The derivation of the time-dependent material functions consists of three steps for the micromechanical analysis of
viscoelastic composites, if use is made of the \textit{Laplace} transform—see Yanzey and Pindera [11]:

\begin{enumerate}
\item \textit{Laplace} transform of the material functions: The material equations in the time domain of all components must be transferred by means of the \textit{Laplace} transform into the image domain. The \textit{Dirichlet–Prony} series in the material function of Eq. (18) transforms into a sum of rational functions.

\[
\tilde{G}(s) = \frac{G_\infty}{s} + \sum_{i=1}^{N} \frac{G_i}{(s + 1/\tau_i)}.
\] (21)

\item Homogenization in the \textit{Laplace} domain: The elastic material equations of the composite in the \textit{Laplace} domain are determined by means of the micromechanical model (see Chapter 2). According to the correspondence principle the elastic material parameters in Eq. (16) are replaced by the \textit{s}-multiple of the \textit{Laplace} transformed material functions \( s\tilde{G}(s) \).

\item Inverse \textit{Laplace} transform: The inverse integral transform from the image into the time domain provides the material functions of the composite. However, the inverse \textit{Laplace} transform can be performed analytically only in a few cases. Usually numerical procedures are necessary for the inversion, where use is made of a \textit{Fourier} series or a \textit{Laguerre} polynomial expansion. The numerical method of D’Amore et al. [12], applied in this investigation, provides the discrete material function at given points in time \( t \) from the analytical function in the \textit{Laplace} domain.
\end{enumerate}

3.2. Numerical time integration

The relaxation integral in Eq. (17) is conveniently solved by numerical time integration as proposed in Taylor et al. [13]. A similar integration procedure has been used in Aboudi [14] in the context of the generalized cell method and viscoelastic materials under finite deformation. The linear structure of Eqs. (8) and (14) is retained also for linear viscoelasticity, but must be adapted to the time stepping scheme. In order to avoid repeated derivations of the discrete equations, the time-dependent equations of the model for flexible fiber–matrix bond are treated first.

4. Modeling of fiber–matrix bond

The influence of a viscoelastic bond between the constituents of the composite is investigated in two models. In the first one a interphase zone with properties different from those of the bulk polymer is examined. In the second approach the time-dependent interfacial flexibility is modeled by a viscoelastic extension of the elastic interface assumption of Jones and Whittier [15]. The equations for the flexible viscoelastic interface may be directly incorporated into the continuity conditions of the cells model. This procedure is computationally very attractive, since it avoids additional cells for the interface between the ones for the matrix and the fiber phase.

4.1. Interphase model with finite thickness

The GMC allows the modeling of arbitrary shaped inclusions in a matrix. Therefore, the interphase region, consisting of the material adjacent to the bond, may by taken into account as a thin layer between the fiber and the matrix phase. The isotropic material behavior, described by the bulk and shear moduli \( K \) and \( G \) of the three phases, can be either elastic or linear viscoelastic. In order to account for the differing interphase properties, two scalars \( \kappa_1 \) and \( \kappa_2 \) are introduced to relate the isotropic interphase properties to the material functions of the matrix:

\[
K_i(t) = \kappa_1 K_m(t),
\] (22)

\[
G_i(t) = \kappa_2 G_m(t).
\] (23)

4.2. Interface models

Flexible bonding effects are advantageously represented by displacement jumps at the fiber–matrix interface with the normal \( \mathbf{n} \). It is assumed that the normal traction vector \( \mathbf{t}_n \) on the interface between the fiber and the matrix is proportional to
the jump of the normal displacement $[u_n]_t$ across the interface. Likewise, the tangential traction $t_t$ should be proportional to the jump of the tangential displacement $[u_t]_t$ across the interface. The displacement vectors in normal and tangential direction are given by

$$[u_n]_t = R_n t_n$$ and $$[u_t]_t = R_t t_t.$$ (24)

where

$$[u_n]_t = u_n n$$ and $$[u_t]_t = u_t - [u_n],$$ (25)

t_n = \sigma_n n, n$$ and $$t_t = t - t_n.$$ (26)

$R_n$ and $R_t$ are the interfacial normal and tangential compliances. A physically unrealistic interpenetration of the matrix phase into the fiber phase is prevented by the constraint $R_n \geq 0$, if the normal stress component is less than zero: $\sigma_n = \sigma_t n, n \leq 0$. The constitutive equations for the interface can be conveniently included into the continuity conditions (8) and the system of Eq. (15) for the method of cells are modified as in Aboudi [6]:

**Interface condition:** $$[A_G + A_R] \langle \varepsilon \rangle = \left[ \begin{array}{c} J \\ 0 \end{array} \right] \langle \varepsilon \rangle.$$ (27)

The components of the additional matrix $A_R$ are functions of the interface parameters $R_n$ and $R_t$ respectively. The effective properties of the composite are determined from Eq. (16). In view of rate-dependent interfacial effects the convolution integrals of the traction history are the viscoelastic counterpart to the elastic compliance in Eq. (24)

$$[u_n(t)] = \int_{-\infty}^{t} R_n(t - \tau) \frac{dt_n(\tau)}{d\tau} d\tau,$$ (28)

$$[u_t(t)] = \int_{-\infty}^{t} R_t(t - \tau) \frac{dt_t(\tau)}{d\tau} d\tau,$$ (29)

where $R_n(t)$ and $R_t(t)$ represent the interface creep function in normal and tangential direction respectively—see Hashin [16]. These equations describe the time influence of the displacement jump in the interface by a linear functional over the creep function of the interface and the stress history. The generalized Kelvin-model may be used to find the creep function of the interface in terms of a finite Dirichlet–Prony series:

$$R(t) = R_0 + \sum_{j=1}^{M} R_j (1 - e^{-t/\xi_j})$$ (30)

with $\xi_j$ as creep times and $R_j$ as spectrum strengths. However, the creep function of the standard linear solid with only one exponential is taken as the material function of the flexible interface for simplicity:

$$R(t) = R_0 + R_1 (1 - e^{-t/\xi}) = R_{\infty} - R_1 e^{-t/\xi}.$$ (31)

The viscoelastic interface relations (28) and (29) can be solved in the context of the micromechanical model equations in two different ways, which are explained for the one-dimensional case, where the traction vector $t$ is replaced by the associated uniaxial stress component $\sigma$ and the displacement vectors $u_n$ and $u_t$ by the uniaxial displacement $u$. 

**4.2.1. Laplace transform and application of viscoelastic correspondence principle**

The constitutive equations in (28) and (29) can be transformed into the Laplace domain as shown before. By using the Laplace-transformed viscoelastic material equation, the displacement jump $[\tilde{u}]$ in the Laplace domain is obtained as a function of the Laplace variable $s$, the transformed interface function $\tilde{R}(s)$ and the relaxation function $\tilde{G}(s)$ of the neighboring material.

$$[\tilde{u}(s)] = s \tilde{R}(s) \tilde{\varepsilon}(s) = s^2 \tilde{R}(s) \tilde{G}(s) \tilde{\varepsilon}(s).$$ (32)

Eq. (27) holds in the Laplace domain, if the time dependency of the compliant interface is considered by Eq. (32) and the viscoelastic material behavior of the components by Eq. (19) in addition to the viscoelastic correspondence principle. The solution of the resulting micromechanical equations for the phase averaged strains $\langle \varepsilon \rangle$ in the Laplace domain and the subsequent numerical inversion into the time domain yield the effective properties of the composite with the viscoelastic interface.
4.2.2. Time domain analysis

The constitutive equation of the micromechanical model with the viscoelastic interface can also be solved within a numerical time domain analysis. The incremental form of Eqs. (28) and (29) must be derived for the time stepping procedure, where the interface model is subjected to the incremental strains in the neighboring material. In the first step the viscoelastic stress response is inserted into Eqs. (28) and (29) under the usual assumption that the material is undisturbed until a time point, identified as zero, i.e. \( t < 0 \). This leads to a composition of two convolution integrals of the form

\[
[u(t)] = \int_0^t R(t - \tau) \left[ \int_0^\tau G(\tau - s) \frac{de(s)}{ds} \, ds \right] \, d\tau.
\]  
(33)

Eq. (33) can be simplified into a single convolution integral by applying the Leibniz rule for the differentiation of a parameter integral

\[
[u(t)] = \int_0^t R(t - \tau) G(0) \frac{de(\tau)}{d\tau} \, d\tau + \int_0^t \int_0^\tau R(t - \tau) \frac{dG(\tau - s)}{d\tau} \frac{de(s)}{ds} \, ds \, d\tau.
\]  
(34)

The sequence of integrations in the second term of the sum on the right hand side may be interchanged:

\[
[u(t)] = \int_0^t R(t - \tau) G(0) \frac{de(\tau)}{d\tau} \, d\tau + \int_0^t \int_0^\tau R(t - \tau) \frac{dG(\tau - s)}{d\tau} \frac{de(s)}{ds} \, ds \, d\tau.
\]  
(35)

After substituting the discrete relaxation function of Eq. (18) and the discrete creep function of Eq. (31) into Eq. (35), the inner integration over \( \tau \) may be performed analytically. The kernel function for the first and second convolution integral become:

\[
F_i(t - \tau) := R(t - \tau) G(0)
= \left( R_\infty - R_1 e^{-t/\tau_i} \right) \left( G_\infty + \sum_{i=1}^N G_i \right),
\]  
(36)

\[
F_2(t - s) := \int_s^t \left( R_\infty - R_1 e^{-\tau/\tau_i} \right) \frac{d}{d\tau} \left( G_\infty + \sum_{i=1}^N G_i e^{-\tau/\tau_i} \right) \, d\tau
= \sum_{i=1}^N \left[ -G_i R_\infty + R_\infty \left( \frac{R_1 \xi}{\tau_i - \xi} \right) e^{-\tau/\tau_i} - \frac{G_i R_1 \xi}{\tau_i - \xi} e^{-\tau/\tau_i} \right]
\]  
(37)

and \( \tau_i \neq \xi \) in the last equation. The constitutive equation for the viscoelastic interface between two viscoelastic phases in a composite turns into the final form:

\[
[u(t)] = \int_0^t F(t - \tau) \frac{de(\tau)}{d\tau} \, d\tau
\]  
(38)

with the new kernel function

\[
F(t) = f_\infty + \sum_{i=1}^N f_i e^{-t/\tau_i} - f_1 e^{-t/\tau_1}
\]  
(39)

and the coefficients

\[
f_\infty := G_\infty R_\infty,
\]

\[
f_i := G_i \left( R_\infty + \frac{\xi R_1}{\tau_i - \xi} \right); \quad \tau_i \neq \xi
\]  
for \( i = 1, \ldots, N \),

\[
f_1 := R_1 \left( G_\infty + \sum_{i=1}^N \frac{\tau_i G_i}{\tau_i - \xi} \right); \quad \tau_i \neq \xi
\]  
for \( i = 1, \ldots, N \).

5. Numerical time integration of rate-dependent constitutive equations

5.1. Viscoelastic interface

The reformulation of Eq. (33) in terms of Eqs. (38)–(42) is necessary for an efficient numerical time integration as proposed in Taylor et al. [13] and Zienkiewicz and Taylor [17]. Eq. (33) may be written at discrete time points. At time \( t = t_n \) it follows: \( ^n \epsilon = \epsilon(t_n) \)
\[ \left[ u^n \right] := \left[ u(t_n) \right] = \int_0^{t_n} \left( f_\infty + \sum_{i=1}^{N} f_i e^{-\frac{\Delta \tau}{\gamma_i}} - f_i e^{-\frac{n+1}{\gamma_i}} \right) \frac{d\varepsilon(\tau)}{d\tau} d\tau, \]  
(43)

and similarly as before:

\[ \left[ u^{n+1} \right] := \left[ u(t_{n+1}) \right] = \int_0^{t_{n+1}} \left( f_\infty + \sum_{i=1}^{N} f_i e^{-\frac{\Delta \tau}{\gamma_i}} - f_i e^{-\frac{n+1}{\gamma_i}} \right) \frac{d\varepsilon(\tau)}{d\tau} d\tau, \]  
(44)

with the auxiliary variables

\[ nq_i := e^{-\frac{n}{\gamma_i}} \int_0^{t_n} e^{\frac{\varepsilon(\tau)}{\Delta \tau}} d\tau, \]  
(45)

\[ np := e^{-\frac{n}{\gamma_p}} \int_0^{t_n} e^{\frac{\varepsilon(\tau)}{\Delta \tau}} d\tau. \]  
(46)

At time \( t = t_{n+1} = t_n + \Delta t \): \( \quad n+1\varepsilon := \varepsilon(t_{n+1}) \)

\[ \left[ u^{n+1} \right] := \left[ u(t_{n+1}) \right] = \int_0^{t_{n+1}} \left( f_\infty + \sum_{i=1}^{N} f_i e^{-\frac{\Delta \tau}{\gamma_i}} - f_i e^{-\frac{n+1}{\gamma_i}} \right) \frac{d\varepsilon(\tau)}{d\tau} d\tau, \]  
(47)

\[ \left[ u^{n+1} \right] = f^{n+1} \varepsilon + \sum_{i=1}^{N} f_i^{n+1} q_i - f_i^{n+1} p \]  
(48)

\[ n+1q_i := \int_0^{t_{n+1}} e^{-\frac{\varepsilon(\tau)}{\Delta \tau}} \frac{d\varepsilon(\tau)}{d\tau} d\tau, \]  
(49)

\[ n+1q_i := \int_0^{t_n} e^{-\frac{\varepsilon(\tau)}{\Delta \tau}} \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{t_n}^{t_{n+1}} e^{-\frac{\varepsilon(\tau)}{\Delta \tau}} \frac{d\varepsilon(\tau)}{d\tau} d\tau, \]  
(50)

\[ n+1p := \int_0^{t_n} e^{-\frac{\varepsilon(\tau)}{\Delta \tau}} \frac{d\varepsilon(\tau)}{d\tau} d\tau + \int_{t_n}^{t_{n+1}} e^{-\frac{\varepsilon(\tau)}{\Delta \tau}} \frac{d\varepsilon(\tau)}{d\tau} d\tau. \]  
(51)

By making use of the group property of exponentials and of Eq. (45), a recursion formula is obtained from Eq. (50) for the stepwise computation of the convolution integral

\[ n+1q_i = e^{-\frac{n}{\gamma_i}} q_i + \Delta n+1q_i. \]  
(52)

with

\[ \Delta n+1q_i := e^{-\frac{\Delta \tau}{\gamma_i}} \int_{t_n}^{t_{n+1}} e^{\frac{\varepsilon(\tau)}{\Delta \tau}} d\tau \]  
(53)

and the initial value:

\[ 0q_i := \varepsilon(0). \]  
(54)

The same holds for the creep term

\[ n+1p = e^{-\frac{n}{\gamma_p}} p + \Delta n+1p \]  
(55)

with

\[ \Delta n+1p := e^{-\frac{\Delta \tau}{\gamma_p}} \int_{t_n}^{t_{n+1}} e^{\frac{\varepsilon(\tau)}{\Delta \tau}} d\tau \]  
(56)

and

\[ 0p := \varepsilon(0). \]  
(57)

The integrals in Eqs. (53) and (56) must be computed numerically for an arbitrary strain history. It is assumed that the time derivative of \( \varepsilon(\tau) \) is constant in the time interval between \( t_n \) and \( t_{n+1} \):

\[ \frac{d\varepsilon(t)}{d\tau} \simeq \frac{1}{\Delta \tau} \left( (n+1)\varepsilon - n\varepsilon \right). \]  
(58)

Eqs. (53) and (56) become:

\[ \Delta n+1q_i = \frac{\tau_i}{\Delta \tau} (1 - e^{-\frac{\varepsilon}{\gamma_i}})(n+1)\varepsilon - n\varepsilon), \]  
(59)

\[ \Delta n+1p = \frac{\tau_p}{\Delta \tau} (1 - e^{-\frac{\varepsilon}{\gamma_p}})(n+1)\varepsilon - n\varepsilon). \]  
(60)

Eq. (47) gives the discretized form of the rate-dependent constitutive relation:

\[ \left[ u^{n+1} \right] = f^{n+1} \varepsilon + \sum_{i=1}^{N} f_i \left[ e^{-\frac{n}{\gamma_i}} q_i + \Delta n+1q_i \right] - f_i \left[ e^{-\frac{n}{\gamma_i}} p + \Delta n+1p \right], \]  
(61)

which may be written in terms of the viscoelastic stiffness of the interface \( S_{int} \) and the history expression \( y_{int} \):

\[ \left[ u^{n+1} \right] = n+1S_{int} \varepsilon + n y_{int}. \]  
(62)
with the definitions

\[ n+1 S_{\text{int}} := f_\infty + \sum_{i=1}^{N} f_i \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \]

\[ - f_r \frac{\xi}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\xi}}\right), \]

\[ n y_{\text{int}} := f_\infty n + \sum_{i=1}^{N} f_i \frac{\xi}{\Delta t} q_i - f_r \frac{\xi}{\Delta t} p \]

(63)

and

\[ \Delta n+1 e := n+1 e - n e. \]

(65)

5.2. Viscoelastic phases and micromechanical relations

The integral equations according to Eq. (17) may be carried over for each stress component in the three-dimensional case of a viscoelastic material

\[ \sigma_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t - \tau) \frac{\partial e_{ij}}{\partial \tau} d\tau. \]

(66)

The stepwise time integration scheme leads to a similar expression for the phase averaged stresses \( \langle \sigma^{(\beta)} \rangle \) in the \( (\beta) \)-cell at time \( t_{n+1} \) as for the displacement jumps in Eq. (62), if a finite \textsc{dirichlet–prony} series is used for the kernel function.

\[ \langle n+1 e^{(\beta)} \rangle = C^{(\beta)}_{e} e^{(\beta)} + \langle \langle y^{(\beta)} \rangle \rangle \]

(67)

with the algorithmic material stiffness \( C^{(\beta)}_{e} \) and the stress history term \( \langle \langle y^{(\beta)} \rangle \rangle \) for a linear viscoelastic material model. Very often the polymeric phases and the glass fibers are approximated by an isotropic elastic bulk term and a rate-dependent isotropic shear behavior.

\[ C^{(\beta)}_{e} := K^{(\beta)} E^{(\beta)} + 2G^{(\beta)} \left[ e^{(\beta)} + \sum_{i=1}^{N} v_i^{(\beta)} \frac{\tau_i^{(\beta)}}{\Delta t} \right] \]

\[ \times (1 - e^{-\frac{\Delta t}{\tau^{(\beta)}}}) \left[ \frac{1}{3} - \frac{1}{3} \otimes 1 \right] \]

(68)

with bulk and shear modulus \( K^{(\beta)} \) and \( G^{(\beta)} \) and contribution factors \( v_i^{(\beta)} \) and the constraint \( v_{\infty}^{(\beta)} = 1 - \sum_{i=1}^{N} v_i \geq 0. \)

\[ \langle n y^{(\beta)} \rangle = 2G^{(\beta)} \left[ v_{\infty}^{(\beta)} \langle \langle e^{(\beta)} \rangle \rangle + \sum_{i=1}^{N} \left( v_i^{(\beta)} e^{\frac{\Delta t}{\tau_i^{(\beta)}}} \langle n q_i^{(\beta)} \rangle \right) \right] \]

\[ + K^{(\beta)} \text{tr} \langle \langle n e^{(\beta)} \rangle \rangle \mathbf{1} \]

(69)

with \( e \) as the strain deviator and similar definitions as above for the current

\[ \langle n q_i^{(\beta)} \rangle := \int_0^{n_\tau} e^{\frac{\Delta t}{\tau_i^{(\beta)}}} \frac{\partial e_i}{\partial \tau} d\tau \]

(70)

and initial value of the auxiliary vector

\[ \langle \langle q_i^{(\beta)} \rangle \rangle := \langle e^{(\beta)} \rangle (0) \].

(71)

The equilibrium conditions (11) and (12) at the interfaces of the viscoelastic subcells with the stresses in the form of Eq. (66) lead to a matrix equation similar to Eq. (14):

\[ A_{n} \Delta^{(n+1)} e_{n} = \langle n y_{d} \rangle, \]

(72)

where the vector

\[ \langle n y_{d} \rangle = \langle \langle y^{(\beta)} \rangle \rangle - \langle \langle y^{(\beta)} \rangle \rangle \]

(73)

is the difference between the history terms at adjacent subcells \( (\beta') \) and \( (\beta') \). The compatibility equations for the displacements at the interfaces of the subcells at time \( t_{n+1} \) read as:

\[ A_{c} \left[ \langle n e_{n} \rangle + \Delta^{(n+1)} e_{n} \right] = J \left( \Delta^{(n+1)} e_{n} \right). \]

(74)

At time \( t_{n} \) the same relationship holds:

\[ A_{c} \langle n e_{n} \rangle = J \Delta^{(n+1)} e_{n}, \]

(75)

which may be subtracted from Eq. (74).

\[ A_{c} \Delta^{(n+1)} e_{n} = J A^{(n+1)} e_{n}. \]

(76)

The discretized displacement jumps of Eq. (61) for the compliant interface are introduced into the compatibility condition in the form of Eq. (7). As before the displacement jumps \( \langle n+1 u_{n} \rangle \) in normal and tangential direction need to satisfy the continuity equations only on an average and not pointwise.

\[ \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left[ \langle n+1 u_{n}^{(\beta')} \rangle \right]_{x'} \left[ x_{\perp}^{(\beta')} \right] \left[ x_{\parallel}^{(\beta')} \right] d\mathbf{x}_{\parallel}^{(\gamma)} \]

\[ = - \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \left[ \langle n+1 u_{n} \rangle \right]_{x'} \left[ x_{\perp}^{(\gamma')} \right] d\mathbf{x}_{\parallel}^{(\gamma')} \]

(77)
The same holds for the interface with its normal in \(x_3\)-direction. Since the boundaries of the subcells are collinear to the coordinate axis, the displacement jump is either in normal or tangential direction and the quantity \([n^{+1}u_i]\) on the right hand side of Eq. (77) must be derived from the appropriate formula in Eqs. (28) or (29). For example, it follows for the compliant interface at \(x_2^0 = h_R/2\) with its normal \(n\) in the direction of the base vector \(i_2\) and with flexibility in the tangential direction \(i_3\):

\[
[u_3^{(β)}(t)] = \int_{-∞}^{t} R_s(t - τ) \frac{d}{dτ}〈u_{32}^{(β)}(τ)〉 dτ, \tag{78}
\]

where

\[
[u_3^{(β)}(t)] = i_3 \cdot [n^{+1}u_i]_s |_{x_2^0 = h_R/2} \tag{79}
\]

and

\[
〈u_{32}^{(β)}〉 = i_3 \cdot 〈t_i⟩ |_{x_2^0 = h_R/2} = i_3 \cdot 〈σ^{(β)}⟩i_2. \tag{80}
\]

The discretized form of the example in Eq. (78) is obtained from the time stepping scheme.

\[
[n^{+1}u_3] = n^{+1}e_3^{(β)} + A_R^{+1}e_3^{(β)} + 〈y_R^{(β)}〉. \tag{81}
\]

The algorithmic interface stiffness \(n^{+1}S_{n31}\) and the history variable \(〈y_R^{(β)}〉\) are defined in Eqs. (63) and (64) respectively, where the interface properties for the tangential compliance \(R_s(t)\) according to Eq. (31) must be inserted. The appropriate jump conditions are substituted on the right hand side into the extended continuity equation (77), which may be cast in matrix form similar to Eq. (8).

At time \(t_{n+1}\):

\[
\begin{align*}
A_G(〈n^e_i⟩ + Δ〈(n^{+1}e_i)⟩) + A_RΔ〈(n^{+1}e_i)⟩ + 〈y_R⟩ & = 〈y_s⟩ + Δ〈(n^{+1}e)⟩. \tag{82}
\end{align*}
\]

The matrix \(A_R\) consists of the algorithmic interface stiffness \(〈S_R^{(β)}⟩\) and the vector \(〈y_R^{(β)}⟩\) of the averaged history variables \(〈y_R^{(β)}⟩\) of the interface.

At time \(t_n\) the compatibility equation with the compliant interface becomes:

\[
A_G(〈n^e_i⟩ + A_RΔ〈n^e_i⟩ + 〈y_R⟩ = 〈y_s⟩ + Δ〈(n^e)⟩. \tag{83}
\]

The difference between Eqs. (82) and (83) is:

\[
(A_G + A_R)Δ〈n^{+1}e_i⟩ = JΔ〈(n^{+1}e)⟩ - 〈y_{RD}⟩ \tag{84}
\]

with

\[
〈y_{RD}⟩ := 〈y_R⟩ - 〈(n^{-1}y_R)⟩ - A_RΔ〈n^e_i⟩. \tag{85}
\]

Eqs. (72) and (84) read in matrix form:

\[
\begin{bmatrix}
A_G + A_R \\
A_M
\end{bmatrix}
Δ〈(n^{+1}e_i)⟩ = \begin{bmatrix}
J \\
0
\end{bmatrix}Δ〈(n^{+1}e)⟩ + 〈−(y_{RD})⟩. \tag{86}
\]

They give the phase averaged strain increments

\[
Δ(〈n^{+1}e_i⟩) = n^{+1}AΔ〈(n^{+1}e)⟩ + 〈y_s⟩ \tag{87}
\]

in terms of the concentration matrix

\[
n^{+1}A := \begin{bmatrix}
A_G + A_R \\
A_M
\end{bmatrix}
^{-1}
\begin{bmatrix}
J \\
0
\end{bmatrix}, \tag{88}
\]

the vector of the history variable

\[
〈y_s⟩ = \begin{bmatrix}
A_G + A_R \\
A_M
\end{bmatrix}
^{-1}
\begin{bmatrix}
−(y_{RD}) \\
−(y_d)
\end{bmatrix}, \tag{89}
\]

and the effective strains increments \(Δ(〈n^{+1}e⟩)\) of the RVE. The phase averaged stresses \(〈n^{+1}σ^{(β)}⟩\) in the subcell \(〈β⟩\) may be computed from Eq. (67) with the known phase strain increments \(Δ(〈n^{+1}e^{(β)}⟩)\), related to the vector of cell strains \(Δ(〈n^{+1}e⟩)\) through Eq. (9). The effective stresses \(〈n^{+1}σ⟩\) of the homogenized continuum are the volume weighted average of the phase stresses, computed from the cell stresses \(〈n^{+1}σ^{(β)}⟩\) over all relevant subcells

\[
〈n^{+1}σ⟩ = \frac{1}{h^l} \sum_{β=0}^{β=N} \sum_{γ=1}^{γ=N} h_β l_γ 〈n^{+1}σ^{(β)}⟩ \tag{90}
\]

as a consequence of Eq. (2).

6. Numerical examples

6.1. Kernel function of interface

The kernel function \(F(t)\) of Eq. (39) combines the time-dependence of the displacement discontinuity across the interface with the strain history of the adjacent material according to Eq. (38). In the special case of the shear relaxation function \(G(t) = 1276 \cdot (0.6 + 0.4e^{−t})\) [MPa] with \(τ_1 = 1.0\ [h]\)
the graphical representation of $F(t)$ is shown for the time-dependent interface compliance $R(t) = 10^{-1} + \alpha_r \cdot 10^{-1} \cdot (1 - e^{-\frac{t}{\xi}}) \text{ [\mu m/GPa]}$ in Fig. 2. The time-varying part of $R(t)$ is controlled by the parameters $\xi$ and $\alpha_r = R_1/R_0$, standing for the retardation time and the contribution factor. The last term governs the relative increase of the creep compliance with time. The kernel functions $F(t)$ is plotted vs. the logarithm of time in the four diagrams of Fig. 2 for fixed values $\alpha_r = 0.1$, 0.25, 0.5, 1.0 and five different creep times in the range from $\xi = 0.1$ [h] to $\xi = 100.0$ [h].

The kernel function $F(t)$ equals the displacement discontinuity $[u(t_n)]$ at time $t = 0$ due to a given strain jump $\epsilon(t)$ at time $t = 0$ in the adjacent material

$\epsilon(t) = \epsilon_0 H(t) = \begin{cases} 0 & \text{for } t < 0 \\ \epsilon_0 & \text{for } t \geq 0 \end{cases}$

with the \textsc{Heavyside}-function $H(t)$. For small retardation times $\xi < \tau_1$—e.g. $\xi/\tau_1 = 0.1$ or $\xi/\tau_1 = 0.5$—creep occurs first and increases the discontinuity $[u]$. As time progresses relaxation of stresses in the neighboring material takes place and reduces the displacement discontinuity. Hence, the growth phase of the kernel function follows a decrease of the kernel. For large creep times $\xi > \tau_1$, say $\xi/\tau_1 = 10.0$ or 100.0, the relaxation part dominates and causes fading stress and subsequently diminishing discontinuities until the creep part prevails. Then the displacement discontinuity at the interface starts to increase.

Fig. 2. Kernel function for different contribution factors $\alpha_r$ and creep times $\xi$. 
For small values of $\varepsilon_\infty$, e.g. $\varepsilon_\infty = 0.1$, the creep effect obviously has little influence, since the time varying capacity of the compliance is small. As $\varepsilon_\infty$ approaches 1.0 the creep influence is large and the longtime value $F(t \to \infty)$ of the kernel is greater than the initial one $F(t = 0) < F(t \to \infty)$. Fig. 2(a)–(d) clearly show that the kernel $F(t)$ may be non-monotonic in contrast to the relaxation or creep function.

6.2. Elastic interface

An elastic compliance of the interface between the elastic fibers and the matrix is considered first. Imperfect bonding conditions are assumed in shear only ($R_t \neq 0$, $R_n = 0$), in normal direction only ($R_t = 0$, $R_n \neq 0$) and in both shear and normal direction ($R_t \neq 0$, $R_n \neq 0$) between the fiber and matrix cells of a 2x2 cell model in the numerical simulation.

The fiber and the matrix shall to be isotropic with the following bulk and shear parameters: $K_f = 47.5$ [GPa], $G_f = 35.6$ [GPa], $K_m = 5.47$ [GPa] and $G_m = 1.69$ [GPa]. For a fixed fiber volume fraction of $c_f = 50.0\%$ the predicted homogenized elastic moduli for the axial stiffness $E_{ax}^*$, axial Poisson’s ratio $\nu_{ax}^*$, transverse stiffness $E_{t}^*$, transverse Poisson’s ratio $\nu_{t}^*$ and the axial shear modulus $G_{ax}^*$ are calculated, normalized and plotted as functions of the interface parameter $R_n$ and $R_t$ in Fig. 3. Both transverse moduli are sensitive with respect to interfacial flexibility, especially in the range between $10^{-2} \leq R \leq 10$ [um/GPa]. However, the axial moduli $E_{ax}^*$ and $\nu_{ax}^*$ are independent of the tangential interface flexibility $R_t$, whereas $G_{ax}^*$ is constant with respect to $R_n$, the interface flexibility in normal direction.

6.3. Viscoelastic interface

The influence of a viscoelastic fiber–matrix bond on the relaxation of the effective axial shear stiffness $G_{ax}^*(t)$ of a homogenized composite is investigated next. The composite consists of a viscoelastic matrix, reinforced by elastic fibers with a volume fraction of $c_f = 50.0\%$. The material parameters of the elastic fibers remain the same as in the example above for the elastic interface. The isotropic linear viscoelastic matrix has a time-independent bulk modulus of $K(t) = 3.56$ [GPa] and a time-dependent relaxation function for the shear modulus of $G(t) = 1.27 \cdot (0.6 + 0.4e^{-t/\tau})$ [GPa] and $\tau_1 = 1.0$ [h].

Fig. 4 shows the relaxation of the effective axial shear stiffness $G_{ax}^*(t)$ of the micromechanical model for perfect bond, for an elastic and for two different viscoelastic interface creep functions $R_s(t)$ with a chosen creep time of $\xi = 10.0$ [h]. The material and the interface properties are chosen in view of a clear demonstration of the mechanical behavior. The interface creep functions $R_s(t)$ are:

\[
R_1^1 = 0,
\]
\[
R_2^1 = 10^{-1}(1.0 - e^{-\frac{t}{10}}) \text{ [um/GPa]},
\]
\[
R_3^1 = 10^{-1}(1.0 - 0.5e^{-\frac{t}{10}}) \text{ [um/GPa]},
\]
\[
R_4^1 = 10^{-1} \text{ [um/GPa]},
\]

whereas perfect bond is assumed with respect to the normal interface flexibility ($R_n = 0$). The model equations are solved independently by numerical time integration and by the method of Laplace transform. Good agreement between the results from the Laplace-transform and the direct time integration can be seen for the relaxation of the effective axial shear stiffness in Fig. 4.

The interface parameter $R_1^1 = 0$ means perfect bonding and the relaxation of the effective stiffness is due only to the viscoelastic matrix. The purely elastic interface causes stiffness reduction of the homogenized substitute continuum and reduces the initial and longtime shear stiffness. The assumption of a purely viscoelastic interface model $R_1^1$ causes the shift of the relaxation curve from the graph of perfect bonding to the one with the elastic interface as expected. The intermediate curve evolves from the supposition of the combined elastic and viscous interface $R_2^1$, which approaches the elastic solution as time tends to infinity.

The course of the normalized displacement discontinuity $\left[\mu(t)\right]/h$ in tangential direction to the interface between cells 11 and 21 over log-time is plotted in Fig. 5 for the four different interface models due to a “suddenly” $(\Delta t = 0.001$ [h]) applied shear strain $(\epsilon_{12}(t)) = 0.01H(t)$ to the RVE.
of Fig. 6. The displacement jump $[u_i(t)]$ is normalized with respect to the size $h$ of the RVE. Obviously, the discontinuity vanishes for perfect bonding $R_i^1$ and monotonically decreases for the purely elastic interface $R_i^4$, since the stresses relax in the matrix and across the interface and, thus, diminish the interface displacement jump. The entirely viscous interface induces a monotonically
growing discontinuity from zero to the asymptotic limit as time progresses—see graph for $R_2^t$. The viscoelastic interface model $R_3^t$ shows a small drop of the displacement jump followed by an increase, which is not surprising when the case of $z_r = 1.0$ in Fig. 2(d) with $\xi = 10.0$ is inspected.

The internal creep and relaxation behavior of the composite with the interface compliance model $R_i^t$ is given in Fig. 7 due to a strain step function $\langle \epsilon_{12}(t) \rangle = 0.01H(t)$ as outlined above. The graphical representation in Fig. 7 of the phase strains of the unit cell shows their time-dependent change. Because of the viscoelastic matrix, the ratio of the instantaneous matrix stiffness to the fiber stiffness decreases as time progresses. This leads at first to an increase of the strain $\epsilon_{12}^{(21)}$ in the matrix subcell 21. The following decrease of the matrix strain component $\epsilon_{12}^{(21)}$ results from the increasing displacement jump $[u_i]$ with time in the viscoelastic interface. The cell strains, weighted with the cell width, and the interface displacement discontinuity add up to the deformation $h\epsilon_{12}$ of the RVE. The width of the fiber cell for 50.0% volume fraction is $h_1 = \sqrt{2}/2$, if the size of the RVE is taken as $h = 1.0$.

The stress relaxation of the homogenized composite is studied for different creep times $\xi = 0.1, 0.99, 10.0$ and 100.0 of the same model as described above with the interface creep function $R_i^t$.
6.4. Viscoelastic interphase model

Alternatively the viscoelastic interphase may be represented by unit cells in the GMC model, which consists of 13 by 13 subcells as shown in Fig. 9. The interphase thickness is assumed as 300.0 [nm]. For typical fiber sizes the ratio of the fiber diameter to the interphase thickness is calculated as 40, which gives a volume fraction of $c_i \approx 6\%$ for the interphase part.

The fibers are considered as elastic with the same moduli as in the previous example of Section 6.2. The matrix shall be viscoelastic with the same parameters as in example 6.3 above. The relaxation curves from the numerical investigation of the interphase parameter $\kappa$ show the influence of a soft ($\kappa < 1$) or a stiff ($\kappa > 1$) value of the scalar $\kappa$ with respect to the transverse viscoelastic modulus $E_i^*(t)$ in Fig. 10 and the axial shear modulus $G_a^*(t)$ in Fig. 11, where the same value $\kappa$ is used for the bulk and the shear multipliers $\kappa = \kappa_1 = \kappa_2$.

The relaxation curve of the effective axial shear stiffness for the perfectly bonded phases ($R^3_{1} = 0$) in Fig. 4 agrees with the relaxation curve for $\kappa = 1.0$ in Fig. 11, since the case $\kappa = 1.0$ represents perfect bonding for the interphase model. The initial effective shear modulus $G_i^*(0)$ and the asymptotic value (longterm shear stiffness) $G_i^*(\infty)$ as time goes to infinity are drawn as a function of the scalar $\kappa$ in Fig. 11. Obviously, as the multiplier $\kappa$ increases, both the initial and the longterm stiffness modulus grow.

The interface and interphase model in examples 6.3 and 6.4 show similar results for the influence of the bonding compliance. A relationship between the interface compliance $R_n$ or $R_t$ and the material...
scalar $\kappa$ may be established with the help of the diagram in Fig. 12, where the initial value of the transverse stiffness $E_t^i(0)$ is plotted versus the interface compliance $R = R_i = R_n$ for the interface model and versus the scalar $\kappa$ for the interphase model.

Fig. 10. Relaxation of effective transverse stiffness $E_t^i(t)$ for various interphase parameters $\kappa [-]$.  

Fig. 11. Relaxation of effective axial shear stiffness $G_a^i(t)$ for various interphase parameters $\kappa [-]$.  

Fig. 12. Comparison of results for transverse stiffness $E_t^i(0)$ and axial shear stiffness $G_a^i(0)$ of interface and interphase models.
As mentioned above, the initial transverse stiffness values $E_t(0)$ for $R_t = 0$ and the one for $\kappa = 0$ agree with each other, since they represent perfect bonding. The effective transverse stiffness $E_t(0)$ is obtained in the case of debonding for $\kappa = 0$ or for $R_t \to \infty$. Both models lead to the same effective stiffness value. Additional pairs of $R$ and $\kappa$ may be determined from the intersection points of the horizontal line for a given value of $E_t(0)$ with both curves. The intersection points give $R$ and $\kappa$, which are plotted as dots in the coordinate chart of Fig. 13. The same procedure is performed for the axial shear stiffness $G_a(0)$ in Fig. 12(b), where the initial value $G_a(0)$ is plotted with respect to $R$ and $\kappa$. The pairs for corresponding values $R$ and $\kappa$ are added to Fig. 13.

Fig. 13. Corresponding parameters $R$ and $\kappa$ leading to the same effective transverse stiffness $E_t(0)$ and effective axial shear stiffness $G_a(0)$.

7. Summary

The GMC was used to study the influence of a viscoelastic fiber–matrix bond on the overall behavior of composite materials in time. The cells model is capable to approximate arbitrary internal microstructures closely. Therefore, an interphase with a finite thickness may be accounted for in a fiber reinforced composite. In addition the elastic interface model according to Jones and Whittier is incorporated into the simulation of imperfectly bonded materials. An extension to a viscoelastic interface model is given in terms of convolution integrals over the past history of the bonding stresses. The constitutive equations for the interface are established in terms of the traction vector on the interface and the displacement discontinuities between neighboring points on opposite sides of the internal boundary. The model equations of the interface are conveniently incorporated into the compatibility equations of the cells model.

The LAPLACE-transformation of the time-dependent material functions and the application of the correspondence principle of linear viscoelasticity to the governing equations of the micromechanical model are used to compute the effective viscoelastic properties of the composite. The results are compared to the solution of the step-by-step time integration of the rate-dependent interface conditions. The numerical simulations of the examples show good agreement between the different approaches.

The influence of a viscoelastic fiber–matrix bond is studied with a viscoelastic interface model between fiber and matrix and compared to the behavior of an interphase with finite thickness in the RVE with three constituents. A diagram shows the relationship between the parameters of different models for the same effective stiffness of the composite.

References


