A Solution Scheme for General Isotropic Elastoplasticity

1. Introduction

A return mapping algorithm is presented for the numerical time integration of the constitutive equations for elastoplasticity with isotropic yield surfaces, constructed from all three invariants of the stress tensor. Based on the first order backward Euler difference formula (BDF) the governing equations for the stresses are solved in the space of the invariants and the discretized persistence parameter. The stresses are recovered afterwards. For a complete outline of the method see [2]. Despite major computational success with the so-called “radial return mapping algorithm” for J2-plasticity - see [1] and [3], a similar solution strategy has not been proposed for general isotropic yield surfaces, where comparable advantage of the isotropic properties can be made as in the case of the Prandtl-Reuss flow theory for the solution of the governing algebraic equations.

2. Governing Equations and Time Discretization

In accordance with classical plasticity the strain tensor \( \varepsilon \) may be decomposed into its elastic and plastic \( \varepsilon^p \) parts. Linear elasticity shall govern the elastic response between stresses and elastic strains: \( \sigma = D(\varepsilon - \varepsilon^p) \) with \( D = K(I \otimes I + 2\mu(I - \frac{1}{2}I)) \) as the isotropic linear elastic constitutive tensor, \( \Pi = \frac{1}{2}[(\varepsilon_{ij}\varepsilon_{kl} + \varepsilon_{ij}\varepsilon_{kl})] \) and \( 1 \otimes 1 = [\delta_{ij}\delta_{kl}] \), respectively \( I = [\delta_{ij}] \), where \( \delta_{ij} \) is the Kronecker symbol. The yield criterion is assumed to be isotropic

\[
F = f(I_1, J_2, J_3) = 0
\]

and may depend on all three stress invariants: \( I_1 = \text{tr} \sigma, \quad J_2 = \frac{1}{2} \text{tr}(SS) \) and \( J_3 = \frac{1}{2} \text{tr}(SSS) \) with \( \text{tr} \) as the “trace-operator” and \( S = \sigma - \frac{1}{3} \text{tr} \sigma \) as the stress deviator. The time discretization of the normality rule for plastic flow \( \varepsilon^p = \gamma \frac{d \gamma}{d \varepsilon} \) provides by means of the Euler backward difference formula in the case of isotropic elastoplasticity the finite difference expression:

\[
\varepsilon^p_{n+1} = \varepsilon^p_n + \lambda \left[ \frac{\partial f}{\partial I_1} 1 + \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3}J_2 1) \right]
\]

with \( \lambda = \gamma_{n+1} - \gamma_n \)

for advancing the inelastic strains from time \( t_n \) to \( t_{n+1} \). The discretized flow equation is substituted into the isotropic elasticity equation:

\[
\sigma^* = \left\{ \sigma + K\lambda \frac{\partial f}{\partial I_1} 1 + 2\mu \lambda \left[ \frac{\partial f}{\partial J_2} S + \frac{\partial f}{\partial J_3} (SS - \frac{2}{3}J_2 1) \right] \right\}_{t_{n+1}}
\]

where \( \sigma^* \) is the trial stress - also called the elastic predictor, defined as \( \sigma^* = D(\varepsilon^p_{n+1} - \varepsilon^p_n) \). Instead of solving the set of six nonlinear equations in (2) together with the yield criterion in (1) for the six unknown stress components and the increment of the plastic multiplier \( \lambda \), (2) is transformed onto the reduced space of its invariants.

3. Solution in the Space of Invariants and Recovery of Stresses

The trial stress tensor in (2) is mapped into its three invariants \( I_1^*, J_2^*, J_3^* \). The notation is simplified by omitting the index \( n + 1 \), since only the stress components at time \( t_{n+1} \) appear in the governing equations besides the trial stresses. In addition, the short hand notation for the partial derivatives \( f_1 = \frac{\partial f}{\partial I_1}, f_2 = \frac{\partial f}{\partial J_2} \) and \( f_3 = \frac{\partial f}{\partial J_3} \) is used at time \( t_{n+1} \) is used. From (2) we construct:

\[
p = \frac{1}{3} \text{tr} \sigma = p^* - K \lambda f_1 \quad \text{and} \quad p^* = \frac{1}{3} \text{tr} \sigma^* = \frac{1}{3} I_1^*
\]

\[
S^* = (1 + 2\mu \lambda f_3) S + 2\mu \lambda f_3 \left( SS - \frac{2}{3}J_2 1 \right)
\]

\[
S^* S^* = (1 + 2\mu \lambda f_3)^2 S S + 2(1 + 2\mu \lambda f_3)(2\mu \lambda f_3) \left( SSS - \frac{2}{3}J_2 1 \right)
\]

\[
+ (2\mu \lambda f_3)^2 \left( SSSS - \frac{4}{9}J_2^2 1 \right)
\]
and from these the two moment invariants:

\[ J_2^2 = \frac{1}{2} \text{tr} (S^* S^*) = \left(1 + 2\mu \lambda f_2 \right)^2 + 2\mu \lambda f_2 \left[ \frac{1}{3} f_3 \right] J_2 + \left(1 + 2\mu \lambda f_2 \right)^2 6\mu \lambda f_3 J_3 \]  

(3b)

\[ J_3^2 = \frac{1}{3} \text{tr} (S^* S^* S^*) = \left(1 + 2\mu \lambda f_2 \right)^2 J_3 + \left(1 + 2\mu \lambda f_2 \right)^2 \left(2\mu \lambda f_3 \right) \left( J_3^2 - \frac{2}{7} J_2^2 \right) \]

(3c)

after use has been made of the Cayley-Hamilton theorem \( S S S = J_2 S + J_3 I \) in order to reduce all polynomial terms of order higher than two. Since practical yield criteria are frequently formulated in the deviatoric plane in terms of polar coordinates, the radial coordinate \( r \) is chosen as \( r = \sqrt{2J_2} = \|S\| \), and the angle coordinate \( \theta \) is identified with the “Lode angle” \( \varphi = \frac{\pi}{3} \). For convenience the variable \( \xi = \cos \theta = \frac{J_3}{(J_2)^{1/3} J_3} \) is introduced. The moment invariants \( J_2^2 \) and \( J_3^2 \) are replaced by \( \rho \) and \( \xi \). Eqs.(1) and (3a) to (3c) yield in terms of the auxiliary functions \( \Lambda = 2\mu \lambda \), \( a = \rho + \Lambda \left( f_2 - \frac{3}{2} \rho f_3 \right) \) and \( b = 3 \frac{\rho}{2} f_3 \), the set of governing equations for the new invariants and \( \Lambda \):

\[
R = \begin{bmatrix}
R_1 \\
R_2 \\
R_3 \\
R_4
\end{bmatrix} = \begin{bmatrix}
I_1^* - 3\rho - \frac{\Lambda}{2\rho} A f_1 \\
2J_2^* - (a^2 + 2ab\xi + b^2) \\
3\sqrt{6} J_3^* - \left[a^2\xi + 3a^2b + 3ab^2\xi + b^3 \left(2\xi^2 - 1\right)\right] \\
f(p, \rho, \xi, \Lambda)
\end{bmatrix} = 0
\]

(5)

The nonlinear equations in (5) are solved for the invariants \( p, \rho \) and \( \xi \) of the stress state and the plastic multiplier increment \( \Lambda \). Except for special cases, e.g. von Mises or Drucker-Prager yield criterion, iterative methods are used to solve the set of nonlinear equations. In the case of \( J_2 \)-plasticity the solution scheme reduces to the so-called “radial return mapping algorithm”, proposed in [1] and [3].

The stress state \( \sigma_{n+1} \) is recovered from the stress deviator \( S \) by means of (4a). Instead of solving the nonlinear equations in (4a) iteratively, block elimination is used to simultaneously solve for the unknowns \( S \) and \( SS \) in (4a) and (4b) after the forth order term \( SSSS \) is reduced by means of the Cayley-Hamilton theorem. This leads to the deviatoric stress \( S \) at time \( t_{n+1} \):

\[ S = \frac{\rho}{a^3 - 3ab^2 - 2b^3} \left[ \frac{\sqrt{6}}{3} b \left( a^2 + 2ab\xi + b^2 \right) 1 + \left( a^2 - b^2 \right) \right] \left( S^* - \sqrt{6} b S^* S^* \right) 
\]

(6)

Eq.(6) reduces to the well-known result \( S = \frac{R_2}{\text{tr}(S S^*)} S^* \) for the von Mises yield criterion \( f_M = p - R_0 = 0 \) with material constant \( R_0 \) and to \( S = \left[ \frac{1 - (f_M)^2}{(1 + (f_M^2))^{1/2}} \right] S^* \) for the Drucker-Prager yield criterion \( f_{DP} = \alpha p + \rho - c_b = 0 \), where \( f_{DP}^* = \alpha p^* + \rho^* - c_b \) with material constants \( \alpha \) and \( c_b \).

4. Concluding Remarks

The number of floating point operations for the solution of the nonlinear equations in the space of invariants is considerably less than for the direct computation of the stresses from the nonlinear equations (1) and (2). The numerical costs may be further reduced, if the yield criterion can be written as the sum of two functions \( f(p, \rho, \xi, \Lambda) = f_a(p) + f_b(\rho, \xi, \Lambda) = 0 \), such that the pressure and deviatoric parts are additively decomposed. The specific algebraic structure of such yield functions allows to decouple the pressure part from the deviatoric part in the tangent matrix for Newton’s method to solve the nonlinear equations in the space of invariants. For details see [2]. Additional hardening variables in general diminish the computational advantage. However, some practical hardening models allow partial elimination of the internal variables during the iterative solution process to regain the small set of coupled, nonlinear equations.

5. References


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